two audiences for whom this book is very useful: Those who want an overview of the current (as of 1985) state of affairs in parabolic equations and those interested in seeing what has been done in the field by Chinese mathematicians.

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Introduction to asymmetric solitary waves, by Juri Engelbrecht. Longman Scientific & Technical and Wiley, New York, 1991, 280 pp., $102.00. ISBN 0-582-07814-8

In 1845 the British engineer J. Scott Russell published results of extensive experiments on a new type of water wave that he had recently discovered. Such waves can be observed without much difficulty in a long tank or gutter of constant rectangular cross-section. Suppose the tank is filled with water to a shallow depth, which is allowed to reach equilibrium, and that a not-too-violent disturbance is then created at one end of the tank. (This may be done, for example, by adding more water behind a barrier and then suddenly removing the barrier.) If conditions are right, one or more smooth humps of water will emerge from the disturbance and propagate down the length of the tank. As the humps gradually separate (those with greater amplitude moving with greater velocity), each will reveal itself to be an individual wave of constant shape and velocity, not dependent on the others for its existence.

The discovery of these so-called solitary waves took contemporary hydrodynamics by surprise; not only had such a phenomenon not been predicted, but theorists had contended that no such waves could exist. By the turn of the century, however, several theoretical studies, which confirmed Scott Russell's observations, had appeared. In particular, Korteweg and de Vries in 1895 presented an interesting derivation of a quantitative expression for a solitary wave. Starting from the full equations of motion for surface water waves and simplifying these equations to reflect the a priori knowledge that the desired solution moves (say) to the right and has amplitude α and wavelength λ related to the water depth h by $(\alpha/h) \approx (h^2/\lambda^2) \ll 1$, they found that a solitary wave would approximately satisfy the partial differential equation

$$u_t + u_x + uu_x + u_{xxx} = 0.$$

(Here $u = u(x, t)$ is the displacement of the water surface from its equilibrium position at location x and time t. Units of time and distance have been scaled to give the equation a simple form.) By substituting the expression $u(x, t) = \Phi(x - Ct)$ into their equation and solving the resulting ordinary differential equation for Φ, Korteweg and de Vries concluded that a solitary wave of speed C could be described by the formula

$$u(x, t) = 3(C - 1)\operatorname{sech}^2(\tfrac{1}{2}(C - 1)^{1/2}[x - Ct]).$$

In the several decades following the first appearance of the Korteweg–de Vries (KdV) model equation, it does not seem to have received much attention—despite the fact that solitary waves themselves remained a topic of lively interest. Perhaps it was felt that the equation had served its purpose in leading to the above formula for solitary waves and that nothing was likely to be gained by looking for other solutions. Indeed, since the equation had been derived under the explicit a priori assumption that the solution resembled a solitary wave, there was no reason to believe that other solutions would be of physical significance.

This state of affairs changed suddenly in the early 1960s, when, through inspired numerical experimentation, Kruskal and Zabusky discovered that the KdV equation actually models the whole process of formation of solitary waves from general initial data. In other words, for a given initial profile $f(x)$, such as might represent the state of the water in our wave tank immediately after the barrier is removed, the solution of the KdV equation which equals $f(x)$ at time zero will evolve into a finite sequence of solitary waves, each propagating to the right with its own velocity. (There will also be an oscillatory "dispersive" component of the solution, which stays to the left of the solitary waves and decays to zero in amplitude.) This result holds even for initial disturbances $f(x)$ which lie far outside the regime for which Korteweg and de Vries' derivation of their model equation is valid. Subsequent experiments by such researchers as Zabusky, Galvin, Hammack, and Segur showed that the qualitative behavior of actual water waves is well described by the KdV equation; for example, the equation accurately predicts the number of solitary waves that will emerge from a given initial disturbance.

When the reversibility in time of the KdV equation is taken into consideration, it follows from the behavior observed by Kruskal and Zabusky that a typical solution of the equation can be thought of as a collection of solitary waves of varying heights, coming from the left at $t = -\infty$, undergoing interactions in

which their individual identities may seem to be lost, and then evolving into a new collection of solitary waves propagating to the right as $t \to +\infty$. There is no obvious reason to expect that the waves that emerge will be related in any simple way to the waves that enter the interaction; but KdV has another surprise in store for us—the same number of waves emerge as enter, with exactly the same amplitudes and velocities! Thus a general KdV solution may be conceived of as composed mainly of a finite number of individual waves, which, though sometimes hidden within an interaction, are present at all times within the solution. The permanent, particle-like nature of these waves led their discoverers to christen them "solitons".

It remained to find a mathematical explanation for the behavior that Kruskal and Zabusky had observed numerically. Apparently the time was ripe for such an endeavor, for the explanation, which was quickly provided, proved to be as unexpected and exciting as the phenomenon it explained. In 1967, Gardner, Greene, Kruskal, and Miura showed that the problem of solving the KdV equation for given initial data could be transformed into the problem of solving a certain *linear* integral equation. (Their novel method of transformation, which used in-depth work done in the 1950s by Gelfand, Levitan, and Marchenko on the classical problem of inverse scattering, goes today by the name of "inverse scattering transform".) This integral equation has simple solutions which, when transformed into solutions of the KdV equation, give explicit representations of the interaction of any given number of solitons, with any given distribution of heights and spatial positions as $t \to -\infty$. In general, such solutions are called multisoliton solutions. A typical example is the solution

$$u(x, t) = 72\left(\frac{4\cosh(2x - 10t) + \cosh(4x - 68t) + 3}{[3\cosh(x - 29t) + \cosh(3x - 39t)]^2}\right),$$

which represents the interaction of a soliton of amplitude 48 and speed 17 with a soliton of amplitude 12 and speed 5.

These discoveries made it clear that Korteweg and de Vries had come across an equation that was more than just a model for solitary waves on water. The simplicity of the equation in combination with the rich structure of its solutions made it a potential paradigm for a type of behavior that might be observed in other physical systems. It had already been observed that the KdV equation itself arose as a model equation for diverse phenomena outside hydrodynamics. Now a search began for other model equations whose solutions might exhibit soliton-like behavior or which might be approachable by methods similar to the inverse scattering transform. The first step in this search was taken by Lax, who in 1968 identified an important algebraic property of the KdV equation which makes the inverse scattering transform feasible. He then could immediately identify a whole hierarchy of partial differential equations with explicit multisoliton solutions. Although none of these equations besides KdV had any known physical significance, it did not take long for Zakharov and Shabat to use Lax's idea to find analogues of multisoliton solutions for the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0,$$

which rivals the KdV equation in the diversity of its applications. There are now quite a few equations of physical interest that have been identified as "soliton equations". Moreover, the study of solitons and associated phenomena now

extends far beyond the confines of mathematical physics into other fields of mathematics, where it has produced many miraculous results (see, e.g., [1–4]).

On the other hand, it is clear that the wider implications of soliton theory are still far from being worked out. For example, although many explicit soliton-like solutions have been found for the Kadomtsev-Petviashvili equation

$$(u_t + u_x + uu_x + u_{xxx})_x + u_{yy} = 0,$$

the role of these solutions in the evolution of general solutions of the equation is still not understood, due to the fact that interactions between two-dimensional solitons are much more complicated than in the one-dimensional case. Even in one dimension, there are many useful model equations, such as the Benjamin-Bona-Mahony equation

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

whose solutions exhibit definite soliton-like behavior that is not explainable by any known theory [5].

The problem of extending the results and concepts of soliton theory to new contexts is the subject of the book under review. Actually, apart from a brief description of the inverse scattering transform for KdV, solitons and soliton theory per se are hardly treated in the book. The emphasis is instead on solitary waves, the term here being interpreted broadly to mean any localized wave that propagates without change of form. In particular, the author is concerned with the problem of deriving suitable model equations for solitary waves in physical systems. He proposes a general method for deriving such equations, which may be summarized as follows. First, identify an underlying hyperbolic equation that describes the wave. Since transport processes in nature occur with finite velocity, it should always be possible to find such an equation, even for diffusive waves (such as heat waves) that are more commonly modeled by parabolic equations. Next, approximate the hyperbolic equation by a simpler model equation by using the a priori assumption that the solution being sought is a solitary wave. (Several schemes for effecting this approximation are described in Chapter 1.) Again, for diffusive or reactive-diffusive waves, the model equation that results from this procedure will in general be different from the standard model equations for reaction-diffusion processes.

This general method of deriving equations is illustrated in the book with two main examples. The first is a derivation of a perturbed KdV equation as a model for seismic waves. Referring to the "perturbed inverse scattering theory", which has been worked out by Karpman and others, the author describes the shape of solitary wave solutions (their profiles are asymmetric, as opposed to the symmetric profiles of KdV solitons) and investigates the growth or decay of the amplitude of these waves as a function of time.

The second, more novel, example is a derivation of the following nonlinear telegraph equation as a model for electrical pulses in nerve fibers:

$$u_{x\xi} + (a + bu + cu^2)u_x + d = 0$$

(where a, b, c, d are constants). More traditionally, nerve pulses are modeled by systems of reaction-diffusion equations such as the Fitz-Hugh/Nagumo (FHN) equations, which are nonlinear of parabolic type. The suggestion of the author is that the above evolution equation may be more amenable to the

techniques of soliton-related mathematics than the standard reaction-diffusion model equations; it is duly pointed out that the form of the evolution equations is reminiscent of that of the familiar soliton equation $u_{x\xi} = \sin(u)$. This line of inquiry, however, is not pursued; rather, the author contents himself here with an evaluation of the suitability of his equation as a model equation. It is found that the equation has traveling wave solutions that resemble experimentally observed pulse waves and that computations of the profiles of such traveling wave solutions are not subject to the instabilities encountered in computing the profiles of FHN traveling waves. (A drawback of the author's model, however, is that unlike the FHN model, it does not predict the existence of pulse waves that vanish smoothly at both extremities.) Also, numerical solutions of the equation are presented which show the evolution of general initial profiles into pulse waves and which demonstrate that the equation successfully models the "threshold effect" (initial disturbances with amplitude below a certain threshold are damped, while those above the threshold are amplified into pulses).

The book is concerned mainly with the art of mathematical modeling and, in particular, with the elaboration of the above-mentioned general method. The reader will find little in the way of theorems or proofs and occasionally will encounter presentations of numerical results whose relevance is difficult to understand (as in the discussion in Chapter 3 of solutions of the perturbed KdV traveling wave equation). The author's case, however, is usually stated quite coherently, and his carefree and easy style makes for pleasurable reading (despite a plethora of misprints). Although the analysis contained in the book is fairly elementary, some familiarity with applied mathematics at the graduate level is presupposed. The reader may find the book helpful as an introduction to some general ideas used in mathematical modeling, and as a guide to the literature for further details.

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