BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 1993

0273-0979/93 $1.00 + $.25 per page

The linear complementarity problem, by R. W. Cottle, J. S. Pang, and R. E. Stone. Academic Press, New York, 1992, xxiv+762 pp., $59.95. ISBN 0-12-192350-9

The linear complementarity problem is a generalization of the problem of finding a solution to a set of simultaneous linear equations. Although the extension is somewhat simple, the resulting problem provides an elegant framework for the theory of linear and quadratic programming and bimatrix games and is sometimes termed the "fundamental problem" of mathematical programming. In the book entitled *The linear complementarity problem*, the authors give the most complete description of the problem to date and provide an excellent reference for any researcher in the field and an interesting and stimulating text for a graduate level course in this area.

For an example of a linear complementarity problem, consider the following solvable linear programming problem:

$$\begin{array}{ll} \text{minimize} & \langle c, x\rangle \\ \text{subject to} & Ax \geq b, \ x \geq 0. \end{array}$$

The standard duality theory for linear programming asserts the existence of a "multiplier" u which satisfies "dual feasibility"

$$A^{\mathrm{T}}u \leq c, \qquad u \geq 0$$

and "complementary slackness" conditions

$$x_i(A^{\mathrm{T}}u - c)_i = 0, \quad u_i(Ax - b)_i = 0, \qquad \text{for all } i.$$

By setting

$$M = \begin{bmatrix} 0 & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} c \\ -b \end{bmatrix}$$

and invoking duality theory, we see that linear programming is equivalent to finding a solution $z = (x, u)$ of the linear complementarity problem

$$w = Mz + q, \qquad w \geq 0, \ z \geq 0, \ \langle z, w\rangle = 0.$$

We shall refer to this problem as $\mathrm{LCP}(M, q)$. Note that the variables w_i and z_i are nonnegative and their products sum up to zero, so that componentwise either $z_i = 0$ or $w_i = 0$—hence, the name "complementary". The scope of the linear complementarity problem, however, is far broader and the solution techniques used for such problems encompass many of those prevalent in mathematical programming. There are many other examples of this form. The first-order optimality conditions for quadratic programming, bimatrix games, market equilibrium, optimal stopping for Markov processes, and finding convex hulls in the plane can all be formulated as LCP's. In fact there are instances of linear complementarity problems arising in the literature since the 1940s; see, for example, the work of Du Val [11]. The formulation of these problems in a complementarity framework enables one to use a unified analysis and adapt

methods for the solution of one type of application to another. In practice, complementarity arises in many places; for example, in power systems, the voltage will have to drop when the power hits an upper bound. Even commercial modeling systems for mathematical programs are now allowing complementarity conditions to be incorporated [2]. A fairly complete treatment of the linear complementarity problem is given in [1] where many of the above examples are described. The emphasis of the book is on the mathematical foundations of the problem and the chapters of the book cover the questions of existence, computation (both iterative and pivotal), geometry, and sensitivity or stability. This book, along with [8], gives a comprehensive (and somewhat complementary) study of the field: for example, the treatment of iterative schemes and degree theory is only covered in [1]; the treatment of complexity issues is only covered in [8]. The book is very well written and contains many challenging exercises at the end of each chapter. Certainly, some of the exercises are so challenging that a "solution manual" would be a valuable aid for using the book as a graduate level text. Also, at the end of each chapter there is a section on historical notes and references; this serves two purposes extremely well. First, it means that the main text is uncluttered by continual references to other work, which makes it very easy to read; second, as a reference, it makes finding the relevant parts of the literature very easy—one just has to look at the end of a particular chapter to find where more information on a particular subject is to be found and also the historical context of the work. I also found many of the comments in these sections to be very illuminating and helpful as well as serving as a historical reference.

Many of the ideas relating to linear complementarity theory can be thought of as simple generalizations of linear equation theory. To understand this, one considers the notion of a "normal cone" to a given convex set X at some point $x \in X$ (which is a generalization of the notion of a normal to a smooth surface). This set consists of all vectors c which make an obtuse angle with every feasible direction in X emanating from x, that is,

$$\{c | \langle c, y - x \rangle \leq 0, \text{ for all } y \in X\}.$$

When $X = \mathbb{R}^n_+$, the nonnegative orthant in $\mathbb{R}^n$, we label this set as $N_+(x)$. Figure 1 gives examples of this set at a general point x and at the origin. A little thought enables one to see that the linear complementarity problem is just the set-valued inclusion

$$0 \in Mx + q + N_+(x), \qquad x \in \mathbb{R}^n_+,$$

or, equivalently, $-(Mx + q) \in N_+(x)$ and $x \in \mathbb{R}^n_+$. If we let X be the whole space $\mathbb{R}^n$, it is easy to see that the normal cone is precisely the set $\{0\}$ and so the "generalized equation" reduces to the system of linear equations

$$0 = Mx + q.$$

Many of the ideas used in linear equation theory can therefore be generalized to linear complementarity theory essentially by using this analogy. Even further generalizations are possible; see, for example, the notion of generalized equations due to Robinson [10].

The first fundamental question one should pose when a new problem has been formulated is when does a solution to the problem exist, and perhaps determine

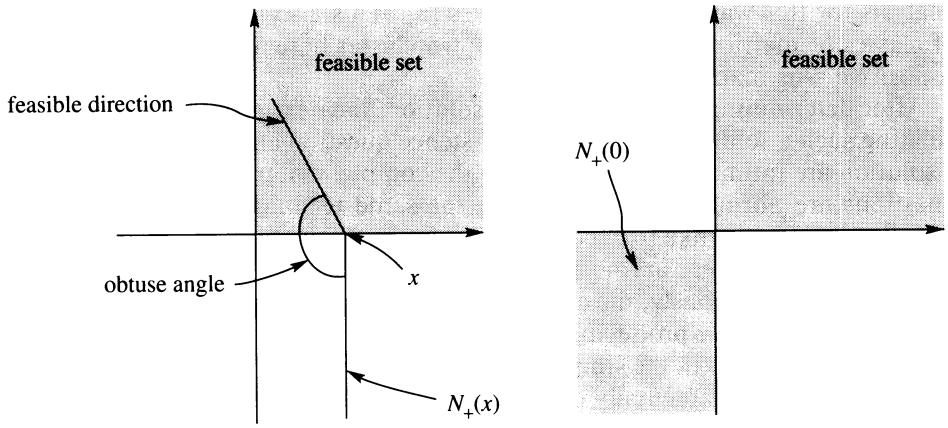

FIGURE 1. THE NORMAL CONE.

the multiplicity of solutions. In the complementarity literature this has led to the investigation of many classes of matrices. Several of the important results (all found in [1, Chapter 3]) are as follows:

- P matrices (those with positive principal minors) characterize the class of linear complementarity problems $\mathrm{LCP}(M, q)$ which are uniquely solvable for all q.
- Positive semidefinite matrices (corresponding to monotone LCP's) are in the class Q_0 of matrices for which feasibility implies solvability.
- Row sufficient matrices are characterized by first-order points for an associated quadratic programming problem solving the linear complementarity problem, and column sufficient matrices are characterized by the solution set of the LCP being convex, for every q.
- Complementarity problems arising from Z matrices, that is, matrices whose off-diagonal elements are nonpositive have a "least element" solution, and this can be determined by solving a single linear program.

The index of [1] has two pages of entries corresponding to different matrix classes! Certainly, an aid to understanding the relationships between the many classes given in this index would be an interesting addition to the book for specialists in the field. The strongest existence results are given in [1, Chapter 3] by means of considering the augmented LCP, $\mathrm{LCP}(\widetilde{M}, \tilde{q})$ defined by

$$\widetilde{M} = \begin{bmatrix} M & d \\ -d^{\mathrm{T}} & 0 \end{bmatrix}, \qquad \tilde{q} = \begin{bmatrix} q \\ \lambda \end{bmatrix}$$

where $d > 0$ is a strictly positive vector. This is very similar to the use of a "Big M" procedure in the simplex method for linear programming, where a problem in a higher-dimensional space is generated that can be processed by the simplex method. Properties of the solution of the linear program as "M" gets larger determines whether the original problem is solvable, generating such a solution if one exists. The augmented LCP always has a solution, even if the original LCP is unsolvable, and the proof of existence of a solution of the original LCP essentially relies on analyzing the behavior of the solutions of these augmented LCP's for unbounded sequences of λ (the "Big M" parameter). The original

analysis for this was carried out by Eaves [3]. This also leads to the definition of classes of matrices: the most well-known classes here are the copositive and copositive plus matrices.

After determining the existence of a solution, the next problem is that of calculating such a solution. Clearly, the existence question and the construction of a solution are intimately related, and in the complementarity field both of these questions are normally answered using the same technique, that is, a pivotal technique due to Lemke [6]. The nice feature of the purely analytical proofs of Chapter 3 is that there are no problems with degeneracy, that is, pivotal steps of zero length. Obviously, the constructive proofs (given in Chapter 4) have the advantage of actually producing solutions. Although the basic step of Lemke's method is a pivot (as in the simplex method for linear programming), the choice of pivot step is fundamentally different and is motivated by a path following or homotopy approach. In linear programming, the incoming variable is chosen to reduce the objective function, the outgoing variable to maintain feasibility. In Lemke's method, there is no objective function. The augmented LCP is used and an "almost complementary" path is generated as follows. Suppose that $\tilde{w} = (w, \gamma)$ and $\tilde{z} = (z, t)$ so that in the augmented LCP above we have

$$\tilde{w} = \widetilde{M}\tilde{z} + \tilde{q}.$$

A "basis" is determined by declaring some of the variables to be dependent on the other variables via this equation. Initially, the basis is "w" and we set $z = 0$ and t sufficiently large. If t can be decreased to zero while maintaining $\tilde{w} \geq 0$, we have a solution to our problem. Otherwise, one of the components of $\tilde{w}$ will hit zero and this variable (the blocking variable) is pivoted out of the basis, being replaced by t. Thus apart from the (t, γ) pair, all the other complementary pairs of variables have their product zero. This is a so-called "almost complementary" point. A path of almost complementary points is generated by forcing the variable complementary to the blocking variable to enter the basis at the next iteration. It follows that this algorithm either terminates at a solution of the original LCP (t becomes the blocking variable) or with a feasible ray for the augmented LCP (no blocking variable exists). Much research has been carried out on the consequences of such ray termination, and this leads to the notion of processing an LCP, which corresponds either to generating a solution or proving that no solution exists. There are many classes of matrices that Lemke's method is known to process. Several of these classes correspond to problems that do not possess underlying convexity properties. It seems that the notion of following a path without enforcing a monotonic decrease in some objective gives rise to more powerful results (note that the value of t in this method can rise or fall). Certainly, in the 1960s and 1970s there was much activity in generalizing these types of methods for finding fixed points of nonsmooth and nonlinear equations. Principal pivoting methods are also covered in Chapter 4, as are computational extensions of both types of pivotal method.

It is possible to look at the LCP from a more geometric standpoint. First, an equivalent formulation of LCP(M, q) is to find a zero of the nonsmooth equation

$$0 = Mx_+ + q + x - x_+$$

where $(x_i)_+ := \max\{x_i, 0\}$ is the projection of x onto the nonnegative orthant.

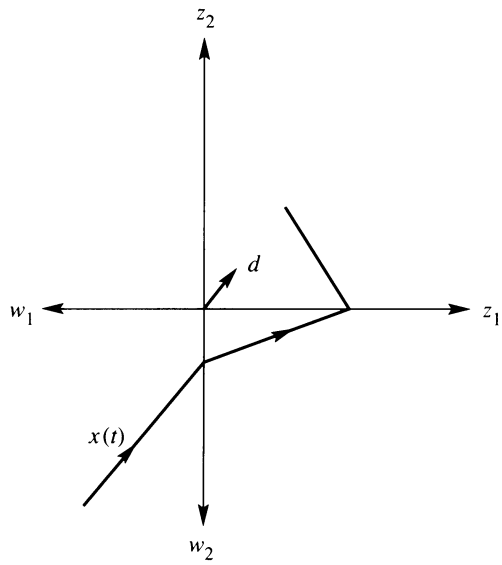

FIGURE 2. THE LEMKE PATH.

This equation is sometimes referred to as the "normal equation", the earliest reference is to [4]. The equivalence is established by noting that if z solves the LCP then $x = z - Mz - q$ satisfies the normal equation, and if x solves the normal equation then $z = x_+$ is a solution of the LCP. It is easy to see that the normal map

$$Mx_+ + q + x - x_+$$

is an affine map on each of the orthants of $\mathbb{R}^n$ and is continuous on $\mathbb{R}^n$. The normal mapping is thus an example of a piecewise affine map and is intimately related to the manifold defined by the collection of the faces of the set $\mathbb{R}^n_+$, called the normal manifold. Lemke's method is now seen to be just a clever way of traversing this manifold. Each pivot step corresponds to a change in the linear map representing the normal map. The vector $d > 0$ is a point in the interior of one of the pieces of the manifold. The almost complementary path corresponds to the points $x(t)$ defined by letting

$$0 = Mx(t)_+ + q + td + x(t) - x(t)_+ .$$

To recover the original scheme, we let $z = x_+$ and $w = x_+ - x$ so that $\langle w, z\rangle = 0$ by definition. The almost complementary variable is again t. In terms of a diagram, essentially we have relabeled the axes in x-space using w and z to arrive at Figure 2. The almost complementary path is also shown. Chapter 6 of the book covers the geometry of Lemke's method in more detail and gives a treatment of the LCP in terms of degree theory. This chapter is most likely to be beyond the reach of the casual reader but contains many new ideas and directions for future research.

If we go back to our motivational example of linear equations, however, there are two broad techniques available for their solution. The first class of techniques are pivotal (namely, Gaussian elimination and its derivatives) and the second class are iterative (namely, Jacobi, Gauss-Seidel, and Conjugate Gradients). While Lemke's method is broadly based on the pivotal techniques, large scale problems require solution using iterative techniques (without explicit factorizations). In equation solving, a standard approach is to let

$$M = B + C$$

and solve the system

$$0 = Bx^{i+1} + Cx^i + q$$

to determine x^{i+1} from x^i (Jacobi, Gauss-Seidel, etc., can be thought of as special instances of this splitting). An analogous splitting for the LCP

$$0 \in Bx^{i+1} + Cx^i + q + N_+(x^{i+1})$$

results in a sequence of problems $\mathrm{LCP}(B, Cx^i + q)$ to solve. The seminal work on iterative approaches for LCP is due to Mangasarian [7], although the use of the terminology of splitting was introduced by Pang [9]. The theory relating to these kinds of splittings is developed in Chapter 5 and is still a subject of much current research in the field. There are three general approaches to proving the convergence of such methods, namely, the symmetry, contraction, and monotonicity approaches. The symmetry approach is useful in the study of convex quadratic programs, the contraction results are used in the study of asymmetric problems, and the monotonicity results look at sequences of iterates which monotonically converge to the solution in a well-specified sense. The symmetry results would appear to be least restrictive of these results (apart from the symmetry assumption of course). A crucial idea here is the notion of a regular splitting, that is, one for which $B - C$ is positive definite and $B + C$ is positive semidefinite. If $B - C$ is positive definite, then it follows that B is positive definite and, hence, that the $\mathrm{LCP}(B, Cx^i + q)$ is uniquely solvable for x^{i+1} [1, Theorem 3.1.6]. The proof of the corresponding convergence result uses error bounds measuring the distance of a given point to the solution set of the complementarity problem. In the contraction approach, the results rest on an eigenvalue analysis and would seem to be far more restrictive. The monotonicity results rely on a hidden Z property of the underlying matrix. An outstanding open question is to derive general conditions that guarantee the convergence of such a matrix splitting scheme when the matrix M is not assumed to be symmetric without using the standard contraction approach.

Due to its close association with linear programming, recent (intense) interest in interior point methods and complexity analysis for linear programs has rubbed off into the linear complementarity field. In fact most of the results have been extended to monotone linear complementarity problems, giving even more justification for considering the LCP as the "fundamental problem". Although §5.9 of the book gives a nice treatment of interior point methods from a nonlinear standpoint, the more standard viewpoint from complexity theory can only be found in recent reserach papers or the books [5, 8]. In fact, much current research pertains to extending these interior point methods to more general classes of linear complementarity problems and to the more general frameworks of nonlinear complementarity and convex programming.

Sensitivity analysis is concerned with how the solution to the LCP changes as one makes changes to the data in M and q. Clearly, in practical situations much of the data is subject to small errors, and we would like to know whether the problem we actually solve will give us any useful information about the problem we are actually modeling. This topic is covered in the last chapter of the book and is couched in the framework of multivalued maps. We have mentioned two such multivalued maps above. One is the normal cone mapping, which takes a given point x and returns a set, the normal cone to the set X at the point x. The second one is the solution set of the complementarity problem: given the data M and q, there is an associated set, which is the solution set of $\mathrm{LCP}(M, q)$. The final chapter of the book looks at the sensitivity analysis by means of the continuity properties of this set-valued mapping. The closely related topic of the parametric linear complementarity problem is also covered in Chapter 4.

There are many more features of the problem which cannot be covered here. An interested reader is referred to [1] for a more complete description of all the ideas outlined here. I would strongly recommend this book as the most up-to-date and complete description of the linear complementarity problem. It is certain to become the standard reference in the field.

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