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My inclination now would be to have a hypothetical student start with this book of Varadarajan, perhaps supplementing the background material with the first three chapters of Sugiura (for an analyst) or the book of Humphreys (for an algebraist), in an independent reading course. Then I would want the student to take a Lie groups course at the level of Varadarajan's text cited above. Finally this student should study material from Helgason, Knapp, Vogan, Wallach, and/or Warner, depending on his level, background, and interests.

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Numerical methods for conservation laws, by Randall J. LeVeque. Birkhäuser-Verlag, Basel, 1992, 214 pp., $24.50. ISBN 3-7643-2723-5

The theory of numerical methods for nonlinear hyperbolic partial differential equations, or conservation laws, has become one of the great successes of numerical analysis. The development of schemes for nonlinear hyperbolic equations requires an understanding of both numerical analysis and the theory of nonlinear hyperbolic equations. By using knowledge of the structure of the solutions of these equations, methods have been developed that compute highly accurate solutions.

The development of the theory of nonlinear hyperbolic partial differential equations in the last fifty years has been stimulated by the growth in applications such as supersonic aerodynamics, thermonuclear explosions, and oil recovery. In each of these applications the differential equations express the conservation of mass, momentum, and other quantities. The increased power of numerical computations has enabled researchers to study ever more complex physical problems governed by conservation laws. A better understanding of the solutions of the differential equations was needed to develop schemes that would compute accurate solutions. Usually the computations of physical phenomena have been beyond the pale of the theory, serving to motivate further work in the theory.

The theory of conservation laws is a rich part of mathematics. One of the best introductions to this theory is the book by Lax [2]. The basic difficulty with nonlinear hyperbolic partial differential equations is that the solutions develop singularities, especially discontinuities, which are usually called shock waves or

shocks. With the development of these discontinuous solutions there is usually the potential for nonuniqueness of the solutions. The mathematical problem is to introduce restrictions on the class of solutions so that a unique solution exists and so that this solution corresponds to physical phenomena. To restrict the set of solutions, it is required that solutions must be weak solutions, that is, they must be consistent with an integral form of the equations. The integral form is obtained by multiplying the differential equation by a test function and then using integration by parts to remove all differentiation from the solution of the differential equation. However, even among the class of weak solutions, the solution is not uniquely determined.

For particular nonlinear hyperbolic systems arising in applications, there usually are physical principles, such as the second law of thermodynamics [1], that select a unique solution from among the weak solutions. In general, the use of similar "entropy conditions" supplies the principle that selects a unique solution from among all weak solutions.

The possibility of nonunique solutions presents difficulties for the development of numerical methods for computing solutions. Indeed, it is easy to construct finite difference schemes for conservation laws that compute incorrect solutions, i.e., solutions not satisfying the correct entropy condition or solutions converging to solutions that are not weak solutions. The basic result on the convergence of solutions of schemes for these equations is the Lax-Wendroff Theorem [3], which states that if a scheme is consistent and conservative, and if the solutions converge as the grid is refined, then the solutions converge to a weak solution of the differential equation. Basically, a conservative scheme is a scheme that satisfies a conservation law analogous to that of the partial differential equation.

The development of the theory of conservation laws for multidimensional problems is an area needing more work. Most of the finite difference schemes currently in use for two-dimensional and three-dimensional computations are simple extensions of one-dimensional schemes to multidimensional problems. These schemes have been quite successful in a wide range of computations, but advances in the mathematical theory are required before better schemes can be developed.

The book by Randall LeVeque is among the first that makes the material in this area accessible to first and second year graduate students in the mathematical sciences. It should be an excellent introduction to this topic for any researcher in the mathematical sciences.

The first half of the book is devoted to the mathematical theory of these equations. There are extensive connections with systems arising from physical applications, especially the Euler equations of fluid flow, the shallow water equations, and the equations of isentropic flow. The book begins with a discussion of linear hyperbolic equations and systems, proceeds to a discussion of nonlinear scalar equations, and then to a discussion of systems. Movement among the topics is smooth and easy.

There is a thorough discussion of the solution of Riemann problems for the different systems. A Riemann problem is one in which the initial data consists of two constant states, one state on the positive real axis and the other on the negative real axis. In keeping with the aim of the book, no general existence theorems for the solutions to the nonlinear partial differential equations are given.

The second half of the book concerns the numerical methods. This begins, as does the first part, with theory for schemes for linear hyperbolic equations followed by the discussion of nonlinear scalar equations.

The chief difficulty associated with constructing schemes for conservation laws is that the schemes must compute correct solutions and also be accurate. The accuracy is especially important near shock waves and other discontinuities.

There are several approaches to the construction of schemes. The simplest methods use "artificial viscosity" to smooth out the discontinuities. It is, however, very difficult to design a smoothing that does not degrade the overall accuracy of the solution. Typically, the choice is between having shocks that are smeared and having oscillations near the shocks. In general, higher-order accurate finite difference schemes for linear hyperbolic partial differential equations generate oscillations near discontinuities and other places where there is difficulty in resolving the solution. These oscillations also occur with schemes for nonlinear hyperbolic equations and cause great difficulty with the jump conditions relating the solutions on either side of the discontinuity.

There are several techniques used to modify schemes from linear equations to nonlinear equations. LeVeque discusses quite thoroughly all the currently useful methods. These include the Godunov methods, flux limiting, and slope limiting. The basic Godunov method is illustrative of one that relies heavily on the partial differential equation theory. In Godunov's method, the discrete solution is regarded as a piecewise-constant function, and the exact solution of the Riemann problem at each discontinuity is used to determine the discrete solution at the next time step. Godunov's method is limited to those systems for which the Riemann problem can be solved. For other systems, and for greater efficiency in general, the exact solution of the Riemann problem can be replaced by the solution to a Riemann problem approximating the exact problem.

Another approach, especially designed to increase the accuracy, is to replace the piecewise-constant solution by piecewise-linear or other piecewise-polynomial solutions. For these approaches, the analog of the Riemann problem cannot be solved exactly, and other approximations are used.

In recent years, several new classes of schemes have been introduced that seek to reduce oscillations without sacrificing accuracy, e.g., the ENO or *essentially nonoscillatory* schemes. These schemes perform quite well in computation, however, the theoretical analysis of these schemes is incomplete. LeVeque gives a nice discussion of the basic ideas of several of these methods.

This book is based on lecture notes of the author and should serve well as a text for a graduate course. The mathematical results that are proved, such as the basic convergence theorems, are well chosen to provide insight. There are many interesting exercises that serve to illuminate and expand on the text, and there are also many well-drawn figures.

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An introduction to multigrid methods, by P. Wesseling. Wiley, New York, 1992, vii+284 pp., $89.95. ISBN 0-471-93083-0

The term *multigrid* refers to a numerical technique that uses a family of grids of differing mesh sizes to discretize and solve continuum problems, most notably partial differential equations. The basic idea is to use a few iterations of an inexpensive *relaxation* scheme (e.g., Gauss-Seidel) on each grid level to attenuate error components of the approximation that vary on a scale comparable to the associated mesh size. These relaxations are performed for equations defined on each grid that approximate a suitable finest-grid error equation (e.g., the residual equation for linear problems), and each result is interpolated to the finest grid to correct the current approximation there. The attraction of these methods is that they are often optimal in the sense that they can produce a result whose accuracy is comparable to the finest-grid discretization error at a cost equivalent to a few finest-grid relaxations. This optimality can often be obtained over a wide range of problem difficulties, including nonselfadjoint operators, nonlinearities, various discontinuities and singularities, and local refinement.

The first known multigrid scheme was developed by Southwell [15] in 1935 for equations of elasticity. Understandably, it was primitive by current standards in its rather awkward use of just two discretization levels. Multigrid schemes using more than two levels were apparently first devised in the 1960s for Poisson's equation by Fedorenko [6] and for more general elliptic equations by Bakhvalov [1], although these methods were still too ineffective to attract much attention. The modern era of truly efficient multigrid techniques was pioneered by Brandt in the 1970s [2, 3]. Since then, multigrid has become the method of choice for a wide range of problems; and its generalizations, referred to by such terms as *multilevel, multiscale*, and *multiresolution*, have begun to invade many diverse disciplines, including aerodynamics, chemistry, civil engineering, economics, geology, image processing, and statistical physics.

Expository publications have lagged well behind technological progress in the multigrid discipline. For several years, the seminal paper by Brandt [3] in 1977 served as the main resource for obtaining an understanding of the practical aspects of multigrid methodology. It contained many of the basic tools and principles that now constitute the core of the discipline, although most of the topics were naturally treated in brief. The "Yellow Book" [8] appeared in 1982 and quickly became the most popular general resource of that decade.