ZARISKI GEOMETRIES

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ABSTRACT. We characterize the Zariski topologies over an algebraically closed field in terms of general dimension-theoretic properties. Some applications are given to complex manifold and to strongly minimal sets.

1. INTRODUCTION

There is a class of theorems that characterize certain structures by their basic topological properties. For instance, the only locally compact connected fields are $\mathbf{R}$ and $\mathbf{C}$. These theorems refer to the classical topology on these fields. The purpose of this paper is to describe a similar result phrased in terms of the Zariski topology.

The results we offer differs from the one considered above in that we do not assume in advance that our structure is a field or that it carries an algebraic structure of any kind. The identification of the field structure is rather a part of the conclusion.

Because the Zariski topologies on two varieties do not determine the Zariski topology on their product (and indeed the topology on a one-dimensional variety carries no information whatsoever), the data we require consists not only of a topology on a set X, but also of a collection of compatible topologies on X^n for each n. Such an object will be referred to here as a geometry. It will be called a *Zariski geometry* if a dimension can be assigned to the closed sets, satisfying certain conditions described below. Any smooth algebraic variety is then a Zariski geometry, as is any compact complex manifold if the closed subsets of X^n are taken to be the closed holomorphic subvarieties.

If X arises from an algebraic curve, there always exist large families of closed subsets of X^2; specifically, there exists a family of curves on X^2 such that through any two points there is a curve in the family passing through both and another separating the two. An abstract Zariski geometry with this property is called *very ample*. By contrast, there are examples of analytic manifolds X such that X^n has very few closed analytic submanifolds and is not ample. Precise definitions will be given in the next section; the complex analytic case is discussed in §4. Our main result is

Theorem 1. *Let X be a very ample Zariski geometry. Then there exists a smooth curve C over an algebraically closed field F, such that X, C are isomorphic as Zariski geometries. F and C are unique, up to a field isomorphism and an isomorphism of curves over F.*

A weaker version is available in higher dimensions (and actually follows easily

from Theorem 1). If X is a higher-dimensional Zariski geometry and there exists a family of curves on X passing through any two points and separating any two, then there exists a dense open subset of X isomorphic to a variety. Globally, X probably arises from an algebraic space in the sense of [A], but we do not prove this. In this survey we will concentrate on the one-dimensional case (though, of course, we must consider the geometry of arbitrary powers).

2. Zariski Geometries

Zariski geometries will be defined as follows. Recall that a topological space is *Noetherian* if it has the descending chain condition on closed subsets. (See [Ha].) A closed set is *irreducible* if it is not the union of two proper closed subsets. If X is Noetherian, then every closed set can be written as a finite union of irreducible closed sets. These are uniquely determined (provided that no one is a subset of the other) and are called the *irreducible components* of the given set. We say that X has dimension n if n is the maximal length of a chain of closed irreducible sets $C_n \supset C_{n-1} \supset \cdots \supset C_0$.

We will use the following notation: if $C \subseteq D_1 \times D_2$, $a \in D_1$, we let $C(a) = \{b \in D_2 : (a, b) \in C\}$.

Definition 1. A Zariski geometry on a set X is a topology on X^n for each n that satisfies the following:

- (Z0) Let f_i be a constant map $(f_i(x_1, \ldots, x_n) = c)$ or a projection $(f_i(x_1, \ldots, x_n) = x_{j(i)})$. Let $f(x) = (f_1(x), \ldots, f_m(x))$. Then $f : X^n \to X^m$ is continuous. The diagonals $x_i = x_j$ of X^n are closed.
- (Z1) Let C be a closed subset of X^n, and let π be the projection to X^k. Then there exists a proper closed subset F of $\mathrm{cl}(\pi C)$ such that $\pi C \supseteq \mathrm{cl}(\pi C) - F$.
- (Z2) X is irreducible and *uniformly 1-dimensional*: if $C \supseteq X^n \times X$ is closed, then for some m, for all $a \in X^n$, $C(a) = X$ or $|C(a)| \leq m$.
- (Z3) (Dimension Theorem) Let U be a closed irreducible subset of X^n, and let Δ_{ij} be the diagonal $x_i = x_j$. Then every component of $U \cap \Delta_{ij}$ has dimension $\geq \dim(U) - 1$.

Axiom (Z0) includes the obvious compatibility requirements on the topologies.

X is called *complete* (or *proper*) if all projection maps are closed. Note that (Z1) is a weakening of completeness. We prefer not to assume completeness axiomatically because we do not wish to exclude affine models.

(Z2) is the assumption of one-dimensionality that we make in this paper. The results still have consequences in higher dimensions, as will be seen in §4.

Axiom (Z3) is valid in smooth algebraic (or analytic) varieties. One still obtains information on other varieties by removing the singular locus, applying the theorem, and going back. It is partly for this reason that it is important not to assume completeness in (Z1).

We now turn to the "ampleness" conditions. By a *plane curve* over X we mean an irreducible one-dimensional subset of X^2. A *family of plane curves* consists of a closed irreducible set $E \subseteq X^n$ (parametrizing the family) and a closed irreducible $C \subseteq E \times X^2$, such that $C(e)$ is a plane curve for generic $e \in E$.

Definition 2. A Zariski geometry X is *very ample* if there exists a family $C \subseteq E \times X^2$ of plane curves such that:

(i) For generic $a, b \in X^2$ there exists a curve $C(e)$ passing through a, b.
(ii) For any $a, b \in X^2$ there exists $e \in E$ such that $C(e)$ passes through just one of a, b.

If only (i) holds, then X is called *ample.*

Axioms (Z0)–(Z3) alone allow a degenerate geometry, in which the closed irreducible subsets of X^n are just those defined by equations of the form $x_i = a_i$. More interesting are the *linear* geometries, where X is an arbitrary field (or division ring) and the closed subsets of X^n are given by linear equations. These are *nonample Zariski geometries.* A thorough analysis of nonample Zariski geometries can be carried out; they are degenerate or else closely related to the above linear example. We will not describe this analysis here; see [HL].

Theorem 1 describes the very ample Zariski geometries, while the nonample Zariski geometries were previously well understood. This leaves a gap—the ample but not very ample Zariski geometries. We can show:

Theorem 2. *Let D be an ample Zariski geometry. Then there exists an algebraically closed field K and a surjective map $f: D \to \mathbf{P}^1(K)$. f maps constructible sets to (algebraically) constructible sets; in fact off a certain finite set, f induces a closed, continuous map on each Cartesian power.*

This represents D as a certain branched cover of $\mathbf{P}^1$. Among complex analytic manifolds, all such covers are algebraic curves. This is not true in the Zariski context; there are indeed ample, not very ample Zariski geometries, which do not arise from algebraic curves. We construct these as formal covers of $\mathbf{P}^1$, starting from any nonsplit finite extension G of a subgroup of $\mathrm{Aut}(\mathbf{P}^1)$ that cannot be realized as the automorphism group of any algebraic curve. We obtain a finite cover of $\mathbf{P}^1$, with an action of G on it, and define the closed sets so as to include the pullbacks of the Zariski closed subsets of $\mathbf{P}^1$ and the graphs of the G-operators. The automorphism group of the cover is then too large to arise from an algebraic curve.

The following is implicit in Theorem 1, but we wish to isolate it:

Theorem 3. *Let K be an algebraically closed field, and let X be a Zariski geometry on $\mathbf{P}^1(K)$, refining the usual Zariski geometry. Then the two geometries coincide.*

These results sprang from two sources, which we proceed to describe.

3. STRONGLY MINIMAL SETS

The original goal was to characterize an algebraically closed field F in terms of the collection of constructible subsets of F^n, rather than in terms of the closed subsets. The motivation was the importance in model theory of certain structures, called strongly minimal sets. (In particular they form the backbone of any structure categorical in an uncountable power (see [BL]).)

Definition 3. A *structure* is an infinite set D together with a collection of subsets of D^n ($n = 1, 2, \ldots$) closed under intersections, complements, projections and

their inverses, and containing the diagonals. These are called the 0-*definable sets*. D is *strongly minimal* if it satisfies:

(SM) For every 0-definable $C \subseteq D^{m+1}$ there exists an integer n such that for all $a \in D^n$, letting $C(a) = \{b \in D : (a, b) \in C\}$, either $|C(a)| \leq n$ or $|D - C(a)| \leq n$.

These axioms can be viewed as analogs of (Z1), (Z2) for the class of constructible sets. Under these assumptions one can prove the existence of a well-behaved dimension theory. In particular, one can state an axiom analogous to ampleness (non-local-modularity; strongly minimal sets *not* satisfying it are well understood). See [Z, HL]. However, it is not clear how to state (Z3) in terms of constructible sets alone. In the absence of such an axiom, it was shown in [Hr1] that the analog of Theorem 2 is false and in [Hr2] that the analog of Theorem 3 also fails.

We note that Macintyre (see [Mac]) characterized the strongly minimal *fields* as the algebraically closed fields. (These are strongly minimal sets with a definable field structure, i.e., the graphs of addition and multiplication are 0-definable subsets of D^3.) A conjecture by Cherlin and the second author that simple groups definable over strongly minimal sets are algebraic groups over an algebraically closed field remains open.

4. COMPLEX MANIFOLDS

Let X be a (reduced, Hausdorff) compact complex analytic space, and consider the topology An on X^n whose closed sets are the closed analytic subvarieties of X^n. Remmert's theorem then implies that the projection of a closed set is closed. It can be shown further that if U is a *locally closed* subset of X^{n+1}, i.e., the difference of two An-closed sets, then the projection of U to X^n is itself a finite union of locally closed sets, or an An-*constructible set*. Thus the An-constructible sets form a *structure* in the sense of §3. If V is a minimal analytic subvariety of X, then V with this structure is strongly minimal; further, if one removes the singular locus of V, one obtains a Zariski geometry in the sense of §1. In this section we discuss some consequences of this observation.

In the analytic context, Theorem 2 resembles Riemann's existence theorem (the part stating that a compact complex manifold of dimension 1 is a finite cover of the projective line). Indeed Riemann's existence theorem would follow from Theorem 2, but the hypothesis of Theorem 2 includes ampleness, which we do not know how to prove directly. Instead we offer a variation in dimensions ≥ 2. Note that the assumption on M is true of a generic complex torus of dimension ≥ 2 (as will also follow from Proposition 3).

Proposition 1. *Suppose M is a compact Kähler manifold of complex dimension ≥ 2, with no proper infinite analytic subvarieties. If $H_1(M) \neq 0$, then M is a complex torus.*

This follows from Theorem 2, as follows. If M were ample, then it would be a finite branched cover of $\mathbf{P}^1(K)$ for some field K, which in the present context also has the structure of a complex analytic space. K must also have complex dimension ≥ 2, which contradicts the classification of the connected locally compact fields cited in the Introduction. Thus M cannot be ample. As mentioned in the previous section, this gives a strong "Abelian" condition; it

is shown in [HP] (in a much more general context) that if ampleness fails in a given geometry, then every closed irreducible subset of a group A supported by that geometry must be a coset of a closed subgroup. We apply this to the Albanese variety A of M ([GH, p. 331]). A fiber of the map from M to A must be finite, or we violate the assumption on infinite analytic subvarieties. The image of M in A is closed and irreducible, hence a coset of a closed subgroup S of A; "closed" here means a complex analytic subvariety, hence a subtorus, of A. By translation we obtain a finite holomorphic map $f : M \to S$. The branch locus of this map gives an analytic subvariety locally of dimension $\dim(M) - 1$ and hence must be empty. Thus M is a finite covering of the complex torus S and hence is itself a complex torus.

Theorem 3 resembles Chow's theorem that a closed analytic subvariety of projective space must be algebraic (see [GH]). Indeed a somewhat more general result can easily be deduced from the theorem.

Proposition 2. *Let X be a complex algebraic variety. View X as a complex analytic space (as in [HA, Appendix 1]). Then any closed analytic subvariety of X is an algebraic subvariety.*

This is easily seen by working with constructible sets, defined to be finite Boolean combinations of closed sets. We have two Zariski geometries on X, given by the algebraic and the analytic structures. X contains smooth algebraic curves, on which again two geometries are induced. By Theorem 1, one obtains the same geometry on each such curve. It follows easily that the two geometries have the same class of constructible sets, so any closed analytic subvariety V of X is algebraically constructible. Then it is easy to see that V must in fact be Zariski closed.

Another application of [HP] yields the following statement:

Proposition 3. *Let $G = C^n / L$ be a complex torus. Then either G has closed analytic subgroups $G_1 \subseteq G_2$, such that G_2 / G_1 is isomorphic to a nontrivial Abelian variety, or the only complex analytic submanifolds of G are subtori and finite unions of their cosets.*

5. METHOD OF PROOF

We begin with a description of the proof of Theorem 2.

1. **A universal domain.** We are given a Zariski geometry X, which we think of as analogous to the Zariski geometry on a curve over an algebraically closed field, and its powers. We wish to find an analog to the finer notion of the K-topology (where K is a subfield of an algebraically closed field); in other words, we wish to have a concept of a closed set "defined over a given substructure K". This cannot be usefully done for X itself; we need to embed X in a larger geometry X^*, a "universal domain". (This is analogous to viewing a number field as a subfield of a larger field of infinite transcendence degree.) This construction of a universal domain is in fact a standard one in model theory; X^* is obtained via the compactness theorem of model theory or by using ultrapowers; see [CK] (saturated models) or [FJ] (enlargements).

2. **A combinatorial geometry.** We assume (1) has been carried out and work directly with the universal domain X^*. An element a of X^* is *algebraically*

dependent on a tuple of elements $b \in X^{*n}$ if there exists a 0-definable closed set $C \subseteq X^{*n} \times X^*$ such that $C(b)$ is finite and $a \in C(b)$. In the case of algebraically closed fields, this coincides with the usual notion. It can be shown in general that this notion of algebraic closure yields a *combinatorial pregeometry*, i.e., it satisfies the exchange axiom: If a is algebraically dependent on $b_1, \ldots, b_n, c$ but not on $b_1, \ldots, b_n$, then c depends on $b_1, \ldots, b_n, a$. Formally, one can define a transcendence basis, dimension, etc. In particular, the *rank* of a subset of X^* is the size of any maximal independent subset thereof.

3. Group configurations.

To construct a field, we will need to find its additive and multiplicative groups. For this purpose we use a general machinery (valid in the strongly minimal context and in fact considerably beyond it) to recognize groups from the trace that they leave on the combinatorial geometry. We will apply this machinery to the affine translation group to obtain the field.

Suppose G is a 1-dimensional group interpretable in the geometry X^*. (Assume for simplicity that the elements of G are points of X^*; the graph of multiplication is assumed to be locally closed.) Let $a_1, a_2, \ldots, b_1, b_2, \ldots$ be generic points of X, i.e., an independent set of elements of X^* (over some algebraically closed base substructure B). Let $c_{ij} = a_i + b_j$. Then $(c_{ij}: i \in I, j \in J)$ forms an array of elements of the combinatorial pregeometry of B-dependence. It is easy to see that the rank of any $m \times n$-rectangle in this array is $m + n - 1$. Moreover, for any permutations σ of I and τ of J, the corresponding permutation of the array arises from an automorphism of the geometry.

Conversely, suppose $(c_{ij}: i, j)$ is an array of elements of the combinatorial pregeometry enjoying the above symmetry property and in which any $m \times n$-rectangle has rank $m + n - 1$. Then one proves the following theorem: There exists a 1-dimensional Abelian group G and independent generic elements a_i, b_j of G, such that c_{ij} and $a_i + b_j$ depend on each other over B. In particular, an infinite Abelian group is involved.

A similar theory is available for a connected group not necessarily Abelian, of any dimension. We note in this connection Weil's theorem on "group chunks" [W]. Weil's theorem allows the recognition of a group from generic data: A binary function which is generically associative and invertible arises from a definable group. Our result is similar but more general. In particular, it permits the function f to be multivalued (i.e., the graph of an algebraic correspondence). (In other words, the hypothesis concerns algebraic rather than rational dependence on certain generic points.) It is shown that given a trace of associativity, there exists a group G such that f is conjugate (by a multivalued correspondence between the given set and G) to the single-valued function $xy^{-1}z$ of G. (See [EH].)

Applying the theorem to the two-dimensional group of affine transformations of a field F, we obtain the following higher-dimensional analog:

Proposition 4. *Let c_{ij} $(i, j = 1, 2, \ldots)$ be a symmetric array of elements of the dependence geometry over B. Suppose every $m \times n =$ rectangle of elements of c_{ij} has rank $2m + n - 2$ $(m, n \geq 2)$. Then there exists an algebraically closed field F defined over B and generic independent elements a_i, b_j, g_k of F, such that c_{ij} and $a_i + b_i g_j$ depend on each other.*

We will merely use the existence of F. We note that if $c_{ij} = a_i + b_i g_j$, then the elements c_{ij} satisfy the relations: $(c_{ij} - c_{ij'})/(c_{i'j} - c_{i'j'}) = (c_{ij} - c_{ij''})/(c_{i'j} - c_{i'j''})$.

4. **Tangency.** So far we have used only the strongly minimal set structure, not the more detailed knowledge of the identity of the closed sets. This comes in via the notion of a specialization. A map f from a subset C of X^* to X^* is said to be a specialization (over B) if for every B-closed set $F \subset C^n$ and all $a_1, \dots, a_n \in C$, if $(a_1, \dots, a_n) \in F$ then $(fa_1, \dots, fa_n) \in F$. Note that no such notion is available for X itself. The properness axiom immediately yields the extension theorem for specializations. The dimension theorem gives a more subtle property, one of whose principal consequences is a "preservation of number" principle: If $C(a)$ is a closed set depending continuously on a, $a \to a'$ is a specialization, and $C(a)$, $C(a')$ are both finite, then there exists a specialization of $\{a\} \cup C(a)$ onto $\{a'\} \cup C(a')$; in particular, the number of points in $C(a)$ cannot go up (but remain finite) under specializations.

This allows us to formalize a notion of intersection multiplicity. Let C_1, C_2 be curves in the "surface" $X \times X$, depending as above on parameters c_1, c_2 (so $C_i = C^i(c_i)$ for some 0-closed set C^i). Let a_1, a_2 be distinct points in the intersection of C_1, C_2. If $(c_1, c_2, a_1, a_2) \to (c_1', c_2', a_1', a_2')$ is a specialization and $a_1' = a_2'$, we say that the specialized curves C_1', C_2' are tangent at a_1'. In general this notion is not intrinsic to C_1', C_2', a_1', but rather depends on the choice of C_1, C_2. It does not always coincide with the usual notion if X is an algebraic curve (and sometimes yields a more fruitful notion, for our purposes).

5. **The field structure.** From the given two-dimensional family of curves, we may obtain a one-dimensional family of curves passing through a single point p_0 in the plane. These curves are considered to be multivalued functions from X to X. As such they can be composed. One would like to identify two curves that are tangent at p_0; intuitively, an equivalence class of curves corresponds to a slope; one then wants to show that composition gives a well-defined operation, which corresponds to multiplication on the slopes and at all events gives a group structure. In practice, tangency is not necessarily an equivalence relation, and a number of other technical problems arise. However, one can find curves c_{ij} such that $c_{ij}^{-1}c_{i'j}$ and $c_{ij'}^{-1}c_{ij'}$ are tangent for all $i < i'$ and $j < j'$. One then shows that this gives an array as in (3), and hence gives an Abelian group structure.

Eugenia Rabinovich observed that in some cases the group obtained in this way is in fact the additive rather than the multiplicative group.

One would like to find the field directly, by considering functions from the plane to itself, modulo tangency, thus interpreting the tangent space to the plane and what should be GL_2 acting on it. This approach poses severe technical problems. Hence, one first interprets an Abelian group as above, then uses it as a crutch to find the field configuration described above. We find curves satisfying, up to tangency, the relations noted following Proposition 4, in which addition is interpreted as the given Abelian group structure and multiplication is interpreted as composition. We show that this gives the field array.

6. **Theorem 3.** *So far we have essentially described the proof of Theorem 2. For Theorem 3 one is given a field F and a Zariski geometry on F, refining the*

usual Zariski topology. One must show that they coincide. If they do not, it can be shown that there exists a plane curve C not contained in any algebraic curve. From this hypothesis one must obtain a contradiction.

We do this by developing an analog of Bezout's theorem. One defines the degree of a curve C to be the number of points of intersection of C with a generic line. Bezout's theorem then states that the number of points of intersection of C with an algebraic curve of degree d is at most $d \cdot \deg(C)$. We give a version of a classical proof of this, "moving" from an arbitrary algebraic curve to a generic one, and from there to a special one for which the result is clear (the unions of d lines). A certain amount of preliminary work is required, showing that the projective plane over F is sufficiently complete for the purposes of such moves by specializations.

Now one considers intersections of C with algebraic curves of high degree d. The number of points of intersection increases linearly with d, but the dimension of the space of curves of degree d increases quadratically; hence, for large d one can find a curve intersecting C in more than $d \cdot \deg(C)$ points. This is only possible if the intersection is infinite and hence contains C, thus contradicting the choice of C above.

Theorem 1 follows from Theorem 2, Theorem 3, and an analysis of covers in the Zariski category. This analysis shows that any ample Zariski geometry X is a cover of a canonical algebraic curve C, such that the pullback of a curve on C^n is typically irreducible in X^n. It follows that if X is very ample, then $X = C$. It may be in general that the richness of the geometry of X arises entirely from that of C, but we do not know a precise statement.

ACKNOWLEDGMENT

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