each other. This is most strikingly the case when so-called mushy regions exist, and these are described in detail in a later chapter, although no real mention is made of their physical interpretation. However, the book ends with a change of emphasis, with descriptions of time-periodic solutions (as might occur, say, with thermostats), approximate solutions to some ingot solidification problems, and some joint work on the wide-open question of alloy solidification, which leads inevitably to the study of vector Stefan problems.[1] The final pages contain some enigmatic statements about the thermodynamic basis for theories of alloy solidification.

I hope it is clear from the above that this book will be extremely valuable to all mathematicians working in free-boundary problems because it collects all the seminal work in the area carried out by the author over the past fifteen years. However, the book does not purport to provide an overview of the subject, for which many conference proceedings or the book of Crank (not even mentioned here) should be consulted. The first-ever English text on the Stefan problem by Rubinstein does receive brief mention, but I was saddened to see a one-dimensional theory described without references to that pioneering work. Indeed, it would have been helpful if the subjectiveness of the book had been stressed in the introduction.

Notwithstanding these criticisms of style, Professor Meirmanov and his translators, Marek Niezgodka and Anna Crowley, are to be congratulated on having produced this admirable record of the mathematical heart of the Stefan problem.

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Hardy-type inequalities, by B. Opic and A. Kufner. Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Longman Group UK Ltd., Harlow, 1990, vii + 333 pp., $44.00. ISBN 0-582-05198-3

In 1920 when G. H. Hardy discovered the inequality

$$\left\{\int_0^\infty \left|\frac{1}{x}\int_0^x f(t)\,dt\right|^p dx\right\}^{1/p} \le \frac{p}{p-1}\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p}, \tag{1}$$

$1 < p < \infty$, in an attempt to simplify the proofs of Hilbert's double series theorem, he could hardly have foreseen the profound influence this inequality and its variants and generalizations would have on the development of many areas in analysis. In Fourier analysis, for example, it is the key factor in the proof of the Hardy-Littlewood maximal theorem; and the proof of the Marcinkiewicz theorem on the interpolation of operators requires, in a significant way, only a

[1]A new kind of vector Stefan problem has recently gained prominence in the macroscopic theory of superconductivity.

mild extension of (1). Indeed, the inequality and its generalizations are some of the basic tools in modern interpolation theory—the real method—and are indispensable in the study of function spaces.

If w and v are weight functions and $Tf(x) = \int_a^x f$, then a weighted generalization of (1) is the inequality

$$(2) \qquad \left(\int_a^b w|Tf|\right)^{1/p} \leq C\left(\int_a^b v|f|^p\right)^{1/p}, \qquad -\infty \leq a < b \leq \infty,$$

where $C > 0$ is a constant independent of f. The question of finding weight conditions which are equivalent to the boundedness and compactness of T on weighted Lebesgue spaces has drawn considerable attention during the last twenty-five years and has now largely been solved. Of course, (2) with $u = Tf$ is also a statement about derivatives, and its n-dimensional analogue raises the following questions: Under what choices of functions u, under what conditions on the domain $\Omega \subset \mathbb{R}^n$, and on the weight functions $w, v_1, v_2, \ldots, v_n$ is the inequality

$$(3) \qquad \left\{\int_\Omega w(x)|u(x)|^q\,dx\right\}^{1/q} \leq C\left\{\sum_{i=1}^n \int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\right|^p v_i(x)\,dx\right\}^{1/p}$$

satisfied? This is the central question discussed by Opic and Kufner in the book under review. Note that for $1 < p < n$, $1 < q < np/(n-p)$, $w = v_1 = v_2 = \cdots = v_n = 1$, and $U \in C_0^\infty(\Omega)$, Ω a bounded domain with Lipschitz boundary, this is Sobolev's inequality; while in the case $p = 1$, $q = n/(n-1)$ with the same choice of function and weights, (3) is known as Gagliardo inequality. Similarly, if $p = q = 2$, Ω a bounded domain where Green's formula holds, one obtains Friedrichs's inequality; while for functions u whose mean value over Ω vanishes, the estimate is called Poincaré inequality. These inequalities then are collectively denoted by the authors as Hardy-type inequalities, and it is shown that they are instrumental in the study of PDEs and weighted Sobolev function spaces.

The monograph begins with a detailed study of the one-dimensional inequality (2) and its dual. It traces the recent history of the inequality carefully and provides the two basically different proofs where weight conditions are shown to be equivalent to (2) with good control on the best constants. One of the weight conditions is based on the solution of a differential equation, while the other is in terms of certain integral conditions which also characterize the inequality in the index range $0 < q < 1$, $p > 1$. Specifically, for $1 < p \leq q < \infty$, the second approach shows that (2) is equivalent to

$$(4) \qquad \sup_{a<x<b} \left(\int_x^b w\right)^{1/q} \left(\int_a^x v^{1-p'}\right)^{1/p'} < \infty,$$

while the condition

$$(5) \qquad \int_a^b \left[\left(\int_x^b w\right)^{1/q} \left(\int_a^x v^{1-p'}\right)^{1/q'}\right]^r v(x)^{1-p'}\,dx < \infty,$$

$1/r = 1/q - 1/p$ is equivalent to (2) in the range $0 < q < p < \infty$, $p > 1$.

It is interesting to note here that (5) is also equivalent to the compactness of the operator $T\colon L_v^p \to L_w^q$ if $1 < q < p < \infty$, although (4) alone is not enough to ensure compactness of T in the index range $1 < p < q < \infty$. Still, in the one-dimensional case, the authors discuss the boundedness of the Riemann-Liouville fractional integral operator on weighted Lebesgue spaces and apply it to obtain weight characterizations for inequalities involving higher-order derivatives.

The characterization of weights in terms of capacity conditions for which (3) holds with $u \in C_0^\infty(\Omega)$ for $1 < p$, $q < \infty$ has already been obtained by Maz′ja and others. In this book the authors are content to provide weight conditions which primarily are only sufficient for (3) and the domains are often special. They are related to the one-dimensional weighted Hardy inequalities and have the advantage that they are easier to verify. In the treatment of continuous and compact embeddings of weighted Sobolev spaces into weighted L^p-spaces, the weights $d(x) = \operatorname{dist}(x, \Omega)$ play a special role. For such weights and their powers, the index conditions for the embeddings are sharp.

The nominal aim of the monograph is to provide a partial survey and to collect recent results regarding Hardy-type inequalities. In this the authors have succeeded. The book is very carefully crafted, providing proofs in great detail and a wealth of representative weight examples. Sometimes it seems, however, that the desire for detail and precision led to an excessive amount of notation and unnecessary typography, which may be distracting for some readers. For example, the introduction of nine notations for certain classes of absolutely continuous functions seems too much, and the repeated writing of constants with their dependence on parameters such as $A_L = A_L(a, b, w, v, p, q)$ (and similarly with A_R, A_R^*, B_L, etc.) unnecessary. These are, however, minor matters and preferable to an overly informal presentation.

While the monograph is not exhaustive in its treatment of Hardy-type inequalities, much material presented here is available only in the primary literature. It is a welcome addition to the literature and can be recommended not only to those who wish to learn about the inequality as it pertains to function theory but also to those who study weighted inequalities in other areas of analysis.

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