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Topics in non-commutative geometry, by Yuri I. Manin. M. B. Porter Lecture Series, Princeton University Press, Princeton, New Jersey, 1991, 164 pp. $35.00. ISBN 0-691-08588-9

Noncommutative geometry emerged as a branch of mathematics at the end of the Grothendieck era. Originally its goal had been to geometrize arbitrary noncommutative rings, i.e., first to associate to a noncommutative ring a "noncommutative spectrum" by extending the construction of the prime spectrum of a commutative ring and then to "glue" these spectra into a "noncommutative scheme". The glueing problem turned out to be very difficult, and it does not seem to have been solved in a way meeting initial expectations. A reason is that general noncommutative spectra are not sufficiently functorial in order to be localized (or glued together) in the usual fashion. However, this is an active field; for an essential recent development see [1]. It is remarkable that, nevertheless, noncommutative geometry has made outstanding progress and has in the last decade been constantly among the "hottest" subjects in pure mathematics as well as in mathematical physics. This is because since the 1970s the very idea of noncommutative geometry has become much more complex. Today this is one of those fascinating subjects which ignore customary interdisciplinary boundaries and where, for instance, algebraic geometry in characteristic p and Feynman integrals live together.

A major development which practically redefined the whole subject was Alain Connes's functional analytic approach to noncommutative geometry. The supply of rings to be "geometrized" here are noncommutative C^*-algebras, but what is crucial is that, bypassing local models, Connes has been able to construct directly K-theoretical and differential geometric invariants of the hypothethical noncommutative spaces. An essential tool he used was has famous noncommutative analog of de Rham's complex. Connes's ideas have proven to be very successful and are dominating today.

Another landmark in noncommutative geometry of the 1980s was Vladimir Drinfeld's construction of quantum groups around 1982. He discovered that earlier work of Ludwig Faddeev and his school on solutions of the Yang-Baxter equation leads to remarkable noncommutative (and noncocommutative) Hopf algebra deformations of the function algebras of classical Lie groups. These new Hopf algebras are the "function rings of quantum groups". Quite quickly quantum groups and their representations became a new field of mathematics, which then expanded and now fully deserves the name quantum geometry. But as in Connes's noncommutative geometry, the emphasis in quantum geometry also shifted away from the original geometrization idea. Here the emphasis seems to be more on "quantizing" commutative geometry, i.e., deforming in a suitable way usual function rings and sheaves over them and then dealing with the deformed objects as if they were functions and sheaves on "quantum spaces". A beautiful example of a subject where quantum geometry merges with Connes-style geometry is quantum tori and Manin's quantized theta functions; see [2; 3, Chapter 6, §4].

Finally, since the mid-1970s there has been a third branch of noncommutative geometry where Grothendieck's methods have been highly successful—supergeometry. Here the rings to be geometrized are $\mathbb{Z}_2$-commutative or supercommutative. Supergeometry was given birth by Felix Berezin, who, inspired by the work of physicists aiming at a "super"-unification of bozons and fermions, realized that this idea has fascinating and far-reaching pure mathematical consequences. Berezin introduced, in particular, an analog of the determinant and a quite funny integration procedure of functions with Grassmann variables, known today respectively as the *Berezinian* and the *Berezin integral*. After the Grothendieck machine was made to work in its full power for supergeometry (by Berezin himself, Dmitry Leites, Yuri Manin, and others), a very nice feature came to the surface; namely, supervarieties can be smooth in spite of the nilpotent functions incorporated in their definition. In other words, when dealing with Grassmann nilpotents instead of usual commutative nilpotents, the standard contradiction between smoothness and nilpotent functions miraculously disappears. Yuri Manin has made fundamental contributions to algebraic supergeometry, one of which certainly is the construction of flag supermanifolds. Flag supermanifolds are as central in algebraic supergeometry as the projective space in algebraic geometry and are crucial in the geometric approach to representation theory to classical Lie supergroups. The Lie superalgebras of the latter have been classified by Victor Kac in his pioneering paper [4].

Manin's recent book targeted by this review is devoted mostly to supergeometry and quantum geometry. It is a collection of topics unified just by the general philosophy of noncommutative geometry and by apparent mathematical beauty. "The choice of material was dictated by personal interests of the author," says

Manin in the preface. The first chapter is an overview (and a very stimulating one) of the whole book, and each of the remaining chapters can be in principle read (or consulted) separately. The book is by no means introductory; another quote: "We have chosen not to explain foundations and first examples, but to develop ... concrete and fairly advanced subjects ... ." This concerns mostly the supergeometric chapters, but to some extent applies also to the rest.

Now I would like to go over the contents. As I previously stated, Chapter 1 is an overview of the book. The first section, "Sources of noncommutative geometry", addresses a broad mathematical audience and explains in an enlightening way how geometry (in particular, algebraic geometry), functional analysis, and physics contributed ideas to the emerging field of noncommutative geometry. I would recommend it to all readers of the *Bulletin* who have not closely followed the development of the subject in the 1980s. The remaining three sections are more technical overviews of major topics in noncommutative geometry. Section 2 reviews some algebraic aspects of Connes's theory including noncommutative de Rham cohomology and cyclic cohomology. Section 3 is entitled "Quantum groups and Yang-Baxter equations". Here Manin introduces quantum groups via Yang-Baxter operators and reviews this approach, which goes back to Faddeev and his collaborators. The last section in this chapter is §4, "Monoidal and tensor categories as a unifying machine". This is a brief introduction into the formalism of monoidal categories with special attention to categories of representations of quasi-triangular Hopf algebras and quasi-Hopf algebras. Manin explains also why monoidal categories can be viewed as a unification base (at least) of quantum geometry and supergeometry.

Chapters 2 and 3 are devoted entirely to holomorphic supergeometry. This is an enormous field which is yet somewhat unevenly explored. The two big topics chosen here by Manin are among the best developed ones and are a good advertisement for the whole theory. Chapter 2 is entitled "Supersymmetric algebraic curves". This is the most complete exposition of this subject of which I know, and it contains many new results. Probably it will serve for some time as a universal reference. First of all, let me note that there is no obvious definition of a "superalgebraic curve". If S is a complex $1|N$ dimensional supermanifold, then a $SUSY_N$-structure on S ($SUSY$-from *su*per*sy*mmetric) is a locally direct locally free subsheaf $\mathcal{T}^N$ of rank $0|N$ in the tangent sheaf $\mathcal{T}_S$ for which the Frobenius form

$$\varphi\colon \bigwedge^2 \mathcal{T}^N \to \mathcal{T}_S/\mathcal{T}^N,$$
$$t_1 \wedge t_2 \to [t_1, t_2] \mod \mathcal{T}^N$$

is nondegenerate and locally has an isotropic direct subsheaf of rank $0|[N/2]$. However, the behaviour of the pairs (S, $SUSY_N$-structure on S) depends essentially on N, and probably only for $N = 1, 2, 3, 4$ are these candidates for the name "supercurve". Manin restricts himself to the consideration of the cases $N = 1$ and $N = 2$. The case $N = 1$ is especially neat. For instance, if $(z|\zeta)$ is a local coordinate system on S, then the sheaf of vector fields of the form $f(z, \zeta)(\partial/\partial\zeta + \zeta\partial/\partial z)$, f being a holomorphic (super)function, can be taken as $\mathcal{T}^1$ and thus defines a local $SUSY_1$-structure on S. In the first section of the chapter Manin introduces the conformal symplectic supergroups $C(2m|N)$ and the projective conformal groups $\mathrm{PC}(2m|N)$ and relates them to the basic

examples of $SUSY_N$-curves: the Riemann superspheres $\mathbb{P}^{1|N}$ $(N = 1, 2)$ and the Lobachevski superplane $(N = 1)$. Next he presents the general definition of a $SUSY_N$-family (a $SUSY_N$-structure on a family of relative dimension $1|N$) and then extends the Schottky uniformization technique to the supercase, constructing, in particular, some beautiful $SUSY_N$-families which he conjectures to have natural universality properties. The following two sections are devoted respectively to automorphic Jacobi-Schottky superfunctions and to superprojective structures on $SUSY_1$-families. In §§5 and 6 Manin discusses the Virasoro and Neveu-Schwarz Lie (super)algebras and, in particular, carries out his construction [5] of the Neveu-Schwarz Lie superalgebras via $SUSY_1$-families. It is based on Alexander Beilinson's and Vadim Schechtman's deep approach to the Virasoro algebra via algebraic curves [6]. The final two sections are devoted to the very intriguing topics of elliptic $SUSY_1$-curves and supertheta-functions. Manin discusses here some very nice results of Andrey Levin which shed light on the still unclear notions of a "superabelian variety" and of a "super-Jacobian variety" and which explain in particular that elliptic $SUSY_1$-families are not algebraic supergroups. This chapter is completed by a short discussion of Igor Skornyakov's results on Π-invertible sheaves on supermanifolds (i.e., locally free sheaves $\mathscr{L}$ of rank $1|1$ with a fixed isomorphism between $\mathscr{L}$ and $\Pi\mathscr{L}$, $\Pi\mathscr{L}$ denoting the sheaf of opposite parity) because of their relevance to the problem of supertheta-functions.

Chapter 3 is entitled "Flag superspaces and Schubert supercells". As I already mentioned, it was Manin who constructed the complex flag superspaces in 1980. Here he just recalls the functors they represent and then goes on to the discussion of Schubert supercells, which are actually his object of study in this chapter. Roughly speaking, the first main result is that the (super)dimensions of Schubert supercells are expressed in terms of the superlength function on the flag Weyl group. For Lie supergroups there is no obvious analog of the Weyl group, and the flag Weyl group is a version of the Weyl group which is very useful for computations with flags in the standard representation but unfortunately lacks some other properties of the Weyl group (and is not defined intriniscially in terms of the Lie superalgebra). Next Manin introduces the schematic closures of Schubert supercells and proves an analog of the classical relation between closures of Schubert cells and order in Weyl groups. These are joint results of Manin and Alexander Voronov [7]. Voronov continued this work by constructing a Bott-Samelson-type desingularization of the closures of supercells [8], which Manin also presents. In the final section of this chapter it is shown that flag superspaces are nothing but factors of the corresponding classical Lie supergroup by arbitrary parabolics, and a description of parabolics in terms of roots is given.

In the fourth and final chapter we are back in the "quantum world" (or rather "quantum superworld" because a $\mathbb{Z}_2$-gradation is assumed also in this chapter). Manin's objective here is to present his method of constructing quantum groups as "automorphism groups" of "quantum (super)spaces". He does this in a quite general setting, extending his earlier results from [9, 10]. However, the underlying idea of this process is so simple and so enlightening that I cannot resist presenting it here (if you are an expert, please skip the next few lines). Assume one replaces the usual 2-dimensional space of vector-columns $\binom{x'}{y'}$ by the space

of "quantum vector columns" $\binom{x}{y}$, i.e., pairs of formal variables x, y satisfying the commutation relation $yx = qxy$, where q is a formal parameter. The question is, could one still interpret the endomorphisms of this space as 2×2-matrices acting on the vector-columns by left multiplication? The answer is yes if one agrees to consider matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries a, b, c, d satisfying the commutation relations $ca = qac$, $db = qbd$, $ad - da = q^{-1}cb - qbc$. The 2×2-matrices which together with their transposes satisfy the above relations are by definition the quantum 2×2-matrices. The quantum group $\mathrm{GL}_q(2)$ of "endomorphisms" of the "quantum space of vector-columns $\binom{x}{y}$" is by definition the algebra $M_q(2)[D^{-1}]$, where $M_q(2)$ is the algebra generated by the entries of quantum 2×2-matrices and $D = ad - q^{-1}bc$ is the quantum determinant. This way of constructing $\mathrm{GL}_q(2)$ differs from Drinfeld's original approach. I learned it first in 1986 at Yuri Manin's seminar in Moscow. Generalizing this idea, Manin constructed in [9] the quantum group of endomorphisms of any "quantum space" corresponding to a quadratic algebra. In this book he presents an even more general construction of the endomorphisms of the quantum (super)space corresponding to any $\mathbb{Z}_2$-graded algebra A with a fixed finite-dimensional space A_1 of generators. As an application of this construction, Manin obtains the multiparameter general linear quantum supergroups, which he had constructed earlier in [11]. He also proves a Poincaré-Birkhoff-Witt theory for them, which, however, holds only for finitely many one-parameter subfamilies. A special section is devoted to the case of regular quantum spaces; these correspond to Gorenstein $\mathbb{Z}$-graded algebras $A = \bigoplus_{m>0} A_m$ with $A_0 = k$ (k being the base field) which are generated by A_1, $\dim A_1 < \infty$, and are of polynomial growth. For $\dim A_1 = 2$ any such algebra turns out to be isomorphic to $k\langle x, y\rangle/(f)$, where either $f = yx - qxy$, $q \neq 0$, or $f = xy - yx - y^2$. The first case corresponds to $\mathrm{GL}_q(2)$ (in the sense that $\mathrm{GL}_q(2)$ is the automorphism quantum group of $k\langle x, y\rangle/(yx - qxy)$ and its dual algebra), and Manin presents the explicit formulas also for the second case. For instance, here the determinant of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ equals $ad - b(c + d) = da - (c + d)b$! Among larger regular algebras Manin gives special consideration to Frobenius algebras A, i.e., algebras such that for some d $\dim A_d = 1$, $\dim A_m = 0$ $\forall m > d$, and the multiplication $A_m \otimes A_{d-m} \to A_d$ is a nondegenerate pairing for all $0 \leq j \leq d$. This chapter is completed by a brief discussion of quantum tori and quantum theta functions (after this book had been completed Yuri Manin wrote his much more detailed paper on quantum theta functions [3]).

I like the book a lot. First, because of the excellent mathematics it contains. But second, because of the truly enlightening way this mathematics gets conveyed to the reader. For many years now Yuri Manin has been generously sharing his insights with the mathematical community and has in this way inspired an amazing amount of research. In particular, a significant portion of the results in this book that he credits to others has been obtained either under his direction or as answers to his questions. I am sure that the book will be a strong catalyst for further research.

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Algorithmic algebraic number theory, by M. Pohst and H. Zassenhaus. Cambridge University Press, Cambridge, 1989, pp. 465, $89.50. ISBN 0-521-33060-2

Suppose that $f \in \mathbf{Q}[T]$ is a polynomial with rational coefficients. If f is irreducible, then the quotient ring $\mathbf{Q}[T]/(f(T))$ is a *number field*, i.e., it is a finite extension of $\mathbf{Q}$. Conversely, it is well known that every number field can be obtained in this way. Multiplying $f(T)$ or T by a rational number if necessary, we may and do assume that $f(T)$ is, in fact, a monic irreducible polynomial with integral coefficients.

The basic arithmetical invariants of the number field $F = \mathbf{Q}[T]/(f(T))$ are its ring of integers, the unit group of this ring, and its ideal class group. Another important invariant, perhaps algebraic rather than arithmetic, is the Galois group G of f or, more precisely, the Galois group of a normal closure of F over $\mathbf{Q}$. The group G is a transitive subgroup of the group of permutations of the roots of f.

In a first course in algebraic number theory it is usually first proved that the ring of integers, i.e., the integral closure R of $\mathbf{Z}$ in F, is a Dedekind ring. This ring is not, in general, a principal ideal ring or even a unique factorization domain. Next, one introduces the *ideal class group* $\mathrm{Cl}(R)$ of R; this is the group of fractional R-ideals modulo the subgroup of principal fractional ideals.