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TOPOLOGICAL INVARIANCE OF INTERSECTION LATTICES OF ARRANGEMENTS IN $\mathbb{CP}^2$

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ABSTRACT. Let $\mathscr{A}^* = \{l_1, l_2, \cdots, l_n\}$ be a line arrangement in $\mathbb{CP}^2$, i.e., a collection of distinct lines in $\mathbb{CP}^2$. Let $L(\mathscr{A}^*)$ be the set of all intersections of elements of A^* partially ordered by $X \leq Y \Leftrightarrow Y \subseteq X$. Let $M(\mathscr{A}^*)$ be $\mathbb{CP}^2 - \bigcup \mathscr{A}^*$ where $\bigcup \mathscr{A}^* = \bigcup\{l_i\colon 1 \leq i \leq n\}$. The central problem of the theory of arrangement of lines in $\mathbb{CP}^2$ is the relationship between $M(\mathscr{A}^*)$ and $L(\mathscr{A}^*)$.

Main Theorem. *The topological type of $M(\mathscr{A}^*)$ determines $L(\mathscr{A}^*)$.*

As a corollary of this, we show that the algebra and homotopy type of $M(\mathscr{A}^*)$ do not determine the topological type of $M(\mathscr{A}^*)$.

Let $\mathscr{A} = \{H_1, \ldots, H_n\}$ be a central arrangement of hyperplanes in $\mathbb{C}^3$. Let $M(\mathscr{A}) = \mathbb{C}^3 - \bigcup\{H_i\colon 1 \leq i \leq n\}$. There is a standard procedure in [8] or [10] for passing from arrangements of hyperplanes in $\mathbb{C}^3$ to arrangements of lines in $\mathbb{CP}^2$. In fact, $M(\mathscr{A}) = M(\mathscr{A}^*) \times \mathbb{C}^*$. The intersection lattice $L(\mathscr{A})$ is the set of all intersections of elements of $\mathscr{A}$ partially ordered by reversed inclusion. [9] shows $L(\mathscr{A})$ completely determines the cohomology ring $H^*(M(\mathscr{A}))$.

This result brings the relation between $L(\mathscr{A}^*)$ and $M(\mathscr{A}^*)$ into focus. An example question: Does $L(\mathscr{A}^*)$ determine the homotopy type, topological type, and diffeomorphic type of $M(\mathscr{A}^*)$? Conversely, do any latter invariants of $M(\mathscr{A}^*)$ determine $L(\mathscr{A}^*)$?

For a general class of projective arrangements in $\mathbb{CP}^2$, we have shown $L(\mathscr{A}^*)$ determines the diffeomorphic type of $M(\mathscr{A}^*)$ [4, 5].

Falk introduced an algebraic invariant for $L(\mathscr{A}^*)$. For two particular projective arrangements in $\mathbb{CP}^2$, he asked if they have isomorphic Orlik-Solomon algebras [1, 2]. Rose and Terao produced such an isomorphism [11, 10]. Then Falk showed the $M(\mathscr{A}^*)$s in his example have the same homotopic type. In view of this example, one would like to ask whether $L(\mathscr{A}^*)$ is determined by the topological type of $M(\mathscr{A}^*)$. The purpose of this note is to announce an affirmative answer to the above question.

Let us restate the main theorem more clearly:

Main Theorem. *Let $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ be two projective line arrangements in $\mathbb{CP}^2$. If $M(\mathscr{A}_1^*)$ is homeomorphic to $M(\mathscr{A}_2^*)$, then $L(\mathscr{A}_1^*)$ is isomorphic to $L(\mathscr{A}_2^*)$.*

In view of the results of Rose-Terao [11] and Falk [3], the following statement follows immediately from the main theorem.

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Corollary. *There exist* $\mathscr{A}_1^*$ *and* $\mathscr{A}_2^*$, *two projective line arrangements in* $\mathbb{CP}^2$ *such that* $M(\mathscr{A}_1^*)$ *and* $M(\mathscr{A}_2^*)$ *have the same homotopy type and isomorphic cohomological algebra, but not the same topological type.*

Because of the main theorem, it makes the first question raised above more interesting. In fact, we believe that the following conjecture is true.

Conjecture. *For any projective line arrangement* $\mathscr{A}^*$ *in* $\mathbb{CP}^2$, *the topological type of* $M(\mathscr{A}^*)$ *is determined by* $L(\mathscr{A}^*)$.

In order to prove the main theorem, we have to separate arrangements in $\mathbb{CP}^2$ into two categories. An arrangement in $\mathbb{CP}^2$ is called *exceptional* if one of its lines has at most two intersection points. An arrangement in $\mathbb{CP}^2$ is called *nonexceptional* if every line in the arrangement has at least three intersection points.

A *regular* neighborhood of an arrangement $\mathscr{A}^*$ in $\mathbb{CP}^2$ can be defined as follows: Choose a finite triangulation of $\mathbb{CP}^2$ in which $\bigcup\mathscr{A}^*$ is a subcomplex. The closed star of $\bigcup\mathscr{A}^*$ in the second barycenter subdivision of this triangulation is then a regular neighborhood of $\mathscr{A}^*$.

The fundamental observation is the following lemma.

Lemma 1. *Let* U_1 *and* U_2 *be regular neighborhoods of* $\mathscr{A}_1^*$ *and* $\mathscr{A}_2^*$ *respectively. If* $M(\mathscr{A}_1^*)$ *is homeomorphic to* $M(\mathscr{A}_2^*)$, *then* ∂U_1 *is homotopic equivalent to* ∂U_2.

Let $\mathscr{A}^*$ be an arrangement in $\mathbb{CP}^2$. Suppose that $\mathscr{A}^*$ has $x_1, \dots, x_k$ ($k \geq 0$) as multiple intersection points (i.e., multiplicity $t(x_i) \geq 3$). By blowing up $\mathbb{CP}^2$ at $\{x_1, \dots, x_k\}$, we get a set $\widetilde{\mathscr{A}^*}$ of lines in a blown-up surface $\widetilde{\mathbb{CP}^2}$. $\widetilde{\mathscr{A}^*}$ is called an *associated* arrangement in $\widetilde{\mathbb{CP}^2}$ induced by $\mathscr{A}^*$. Each pair of lines of $\widetilde{\mathscr{A}^*}$ intersects at most one point. Let $U(\widetilde{\mathscr{A}^*})$ be a regular neighborhood of $\widetilde{\mathscr{A}^*}$ and $K(\widetilde{\mathscr{A}^*}) = \partial U(\widetilde{\mathscr{A}^*})$. Thus $K(\widetilde{\mathscr{A}^*})$ is a plumbed 3-manifold which is homeomorphic to $K(\mathscr{A}^*)$, the boundary of a regular neighborhood of $\mathscr{A}^*$ in $\mathbb{CP}^2$.

A class of 3-manifolds was classified by Waldhausen [12] in terms of *graphs* and *reduced graph structures* of 3-manifolds. We call these 3-manifolds classified in [12] as *Waldhausen graph manifolds.*

Lemma 2. *If* $\mathscr{A}^*$ *is a nonexceptional arrangement in* $\mathbb{CP}^2$, *then* $K(\widetilde{\mathscr{A}^*})$ *is a Waldhausen graph manifold.*

We define a graph $G(\widetilde{\mathscr{A}^*})$ of $\widetilde{\mathscr{A}^*}$ as follows. Let each vertex correspond to a line in $\widetilde{\mathscr{A}^*}$ with the weight of the self-intersection number of this line. Let each edge correspond to the intersection point of two lines in $\widetilde{\mathscr{A}^*}$.

We state some definitions and results derived from [12, 13]. Let M and N be compact orientable 3-manifolds. An isomorphism ψ of $\pi_1(N)$ onto $\pi_1(M)$ is said to *respect the peripheral structure* if for each boundary surface F in N there is a boundary surface G of M such that $\psi(i_*(\pi_1(F))) \subset R$ and R is conjugate in $\pi_1(M)$ to $i_*(\pi_1(G))$ where i_* denotes inclusion homomorphism.

Theorem 3 (cf. [13, (6.5)]. *If* M *and* N *are two Waldhausen graph manifolds and* ψ *is an isomorphism from* $\pi_1(N)$ *onto* $\pi_1(M)$ *which respect the peripheral structure and* $H_1(M)$ *is infinite, then there exists a homeomorphism from* N *to* M *which induces* ψ.

Theorem 4 (cf. [12, (9.4)]). *Two Waldhausen graph manifolds are homeomorphic if and only if the corresponding graphs are equivalent.*

Now suppose that $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ are two nonexceptional arrangements in $\mathbb{CP}^2$ and $M(\mathscr{A}_1^*)$ is homeomorphic to $M(\mathscr{A}_2^*)$. In view of Theorem 3 and Lemmas 1 and 2, we have that $K(\widetilde{\mathscr{A}_1^*})$ is homeomorphic to $K(\widetilde{\mathscr{A}_2^*})$. By Theorem 4 we conclude that there is an isomorphism from $L(\widetilde{\mathscr{A}_1^*})$ to $L(\widetilde{\mathscr{A}_2^*})$. This isomorphism also preserves weights (i.e., self-intersection number). So the main theorem follows from

Theorem 5. *Let $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ be two arrangements in $\mathbb{CP}^2$. By blowing up their multiple points (of multiplicity ≥ 3), we obtain two associated arrangements $\widetilde{\mathscr{A}_1}^*$ and $\widetilde{\mathscr{A}_2}^*$ in some blown-up surfaces $\widetilde{\mathbb{CP}^2}$. Then there exists an isomorphism from $\widetilde{L}(\mathscr{A}_1^*)$ onto $L(\widetilde{\mathscr{A}_2^*})$ which preserves weights if and only if there is an isomorphism from $L(\widetilde{\mathscr{A}_1^*})$ onto $L(\widetilde{\mathscr{A}_2^*})$.*

We next suppose that both $\mathscr{A}_1^*$ and $\mathscr{A}_2^*$ are exceptional. Write

$$\mathscr{A}_1^* = \{H_0, H_1, \dots, H_p, \ H_{p+1}, \dots, H_{p+q}\}, \tag{1}$$

$$\mathscr{A}_2^* = \{H_0, G_1, \dots, G_s, \ G_{s+1}, \dots, G_{s+t}\} \tag{2}$$

where H_0 (respectively G_0) intersects with $H_1, \dots, H_p$ (respectively $G_1, \dots, G_s$) at one point and interacts with $H_{p+1}, \dots, H_{p+q}$ (respectively, $G_{s+1}, \dots, G_{s+t}$) at another point. If $M(\mathscr{A}_1^*)$ is homeomorphic to $M(\mathscr{A}_2^*)$, then the Orlik-Solomon algebras associated to A_1 and A_2 are isomorphic. It follows that $p+q=s+t$ and $pq=st$. So $L(A_1)$ is isomorphic to $L(A_2)$.

Finally, we assume that $\mathscr{A}_1^*$ is exceptional, but $\mathscr{A}_2^*$ is not. We need to show that $M(\mathscr{A}_1^*)$ is not homeomorphic to $M(\mathscr{A}_2^*)$. There are four subcases to consider.

Case a. $\mathscr{A}_1^*$ consists of at most three lines. We need to observe only that the first betti number of $M(A)$ is precisely $|A|$. So we have $b_1(M(A_1)) \leq 3 < b_1(M(A_2))$, and $M(\mathscr{A}_1^*)$ is not homeomorphic to $M(\mathscr{A}_2^*)$.

Case b. $\mathscr{A}_1^*$ is a pencil, and $|\mathscr{A}_1^*| \geq 4$. This follows immediately from the following two lemmas.

Lemma 6. *Let $\mathscr{A}^*$ be an arrangement in $\mathbb{CP}^2$. If $\mathscr{A}^*$ is not a pencil (i.e., $\bigcap \mathscr{A}^* = \varnothing$) and $|\mathscr{A}^*| \geq 3$, then $b_3(M(\mathscr{A}))$, the third betti number of $M(\mathscr{A})$, is nonzero.*

Lemma 7. *Let $\mathscr{A}^*$ be an arrangement in $\mathbb{CP}^2$. If $\mathscr{A}^*$ is a pencil (i.e., $\bigcap \mathscr{A}^*$ is a point), then $b_3(M(\mathscr{A}))$, the third betti number of $M(\mathscr{A}^*)$, is zero.*

Case c. $\mathscr{A}_1^*$ consists of a pencil and a line in general position, and $|A_1^*| \geq 4$ (see Figure 1).

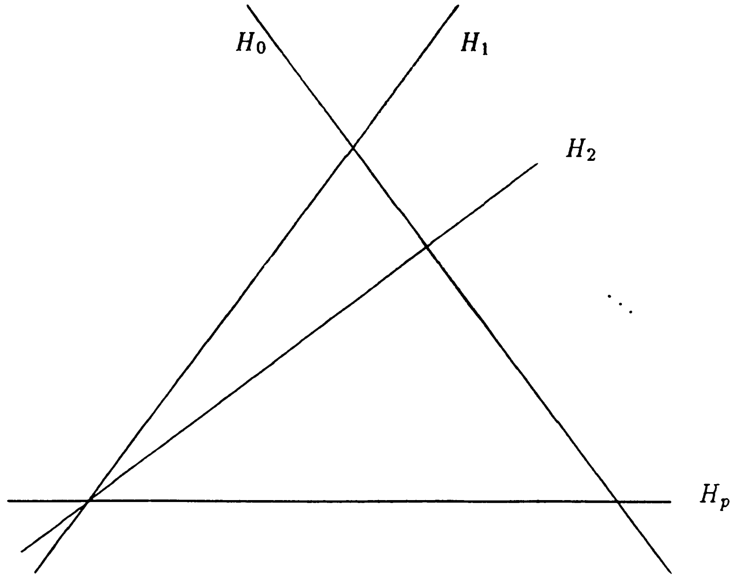

FIGURE 1

By using Neumann's calculus of plumbing [7], one can show that $K(\mathscr{A}_1^*)$, the boundary of a regular neighborhood of $\mathscr{A}_1^*$, is a reduced graph manifold with reduced graph structure equal to empty set. It follows from Lemma 1 and Theorems 3 and 4 that $M(\mathscr{A}_1^*)$ is not homeomorphic to $M(\mathscr{A}_2^*)$.

Case d. $\mathscr{A}_1^* = \{H_0, H_1, \dots, H_p, H_{p+1}, \dots, H_{p+q}\}$ where $\bigcap_{i=0}^{p} H_i$ and $H_0 \cap (\bigcap_{i=p+1}^{p+q} H_i)$ are two different nonempty intersections, $p > 1$ and $q > 1$ (see Figure 2).

By blowing up the points $\bigcap_{i=1}^{p} H_i$ and $\bigcap_{j=p+1}^{p+q} H_j$, we get the following pictures (Figure 3) where E_1 and E_2 are exceptional lines.

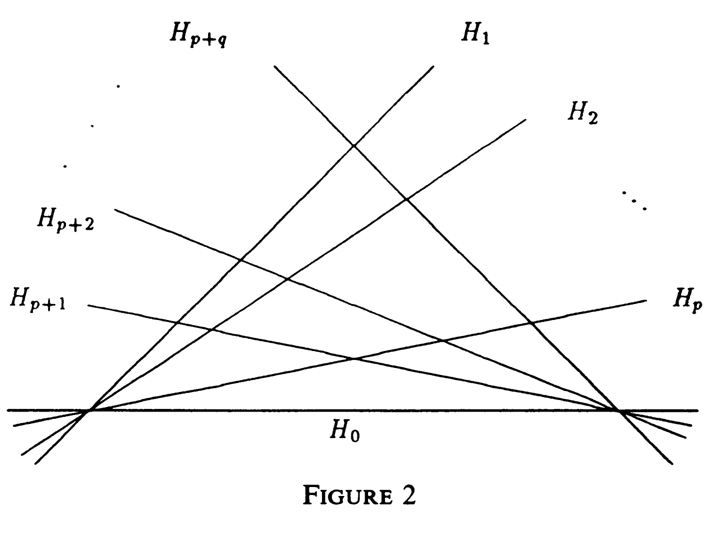

FIGURE 2

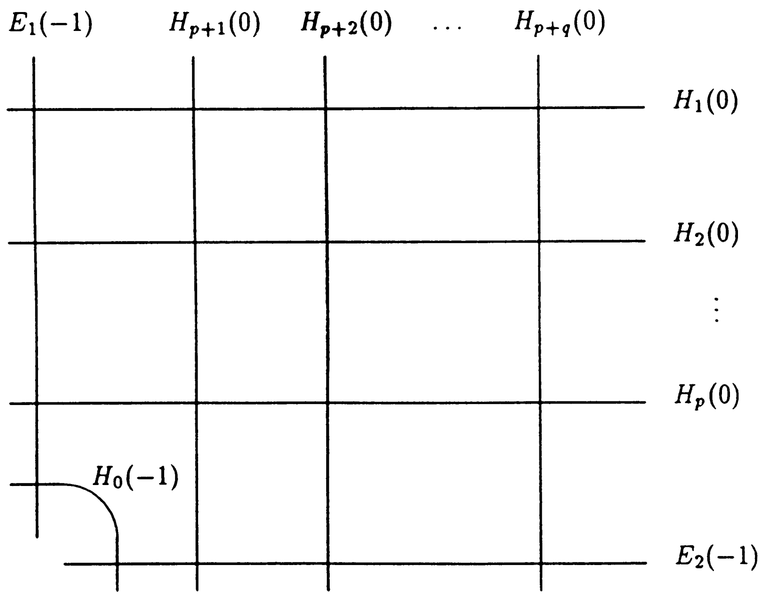

FIGURE 3

Here the numbers in the parenthesis are the self-intersection numbers. We blow down H_0 to a point and get the following pictures (see Figures 4 and 5).

The graph manifold $K(\mathscr{A}_1^*)$ is then a Waldhausen graph manifold with the graph G. Notice that G has only zero weights, while the graph of $K(\mathscr{A}_2^*)$ has nonzero weight. Therefore, by Lemma 1 and Theorems 3 and 4 again, we know that $M(\mathscr{A}_1^*)$ is not homeomorphic to $M(\mathscr{A}_2^*)$.

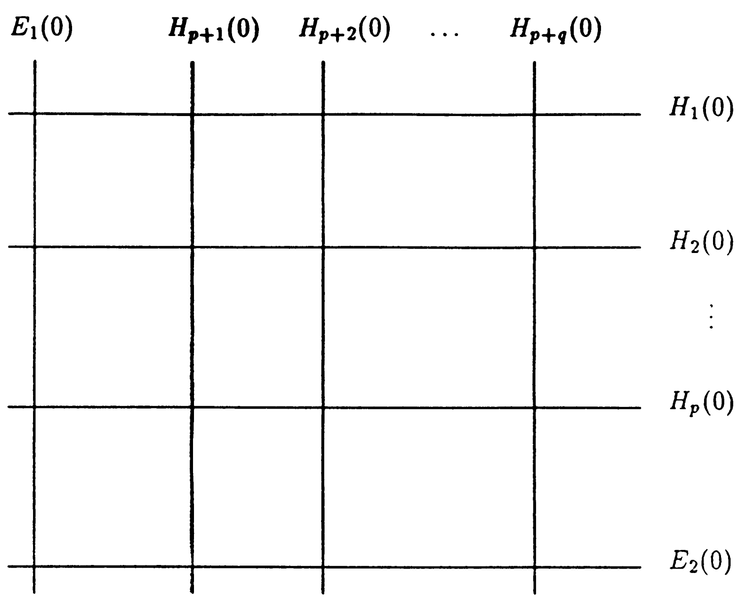

FIGURE 4

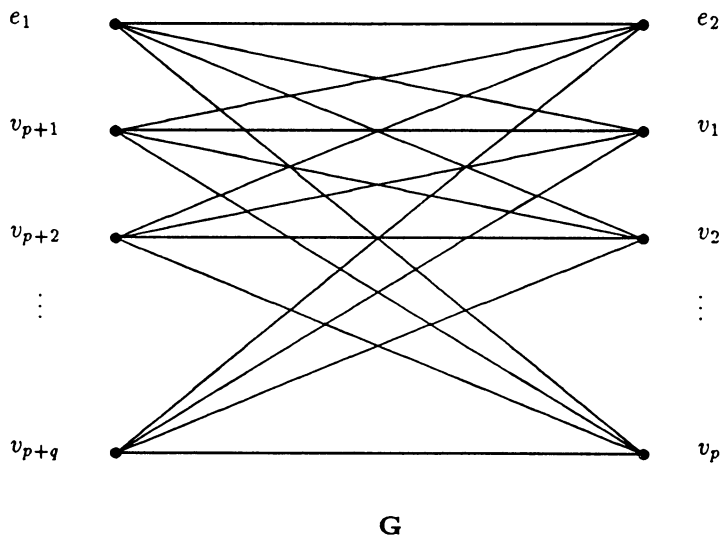

FIGURE 5

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