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Pisot and Salem numbers, by M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J. P. Schreiber. Birkhäuser, Basel, 1992. ix + 291 pp., $124.00. ISBN 3-7643-2648-4

The book under review is not the first one that discusses Pisot and/or Salem numbers. Among previous books one should mention [2, 3, 5, 6]. This book is quite different in spirit: unlike the others, it is more concerned with purely number theoretical problems (both analytic and algebraic). The last chapter (Chapter 15) does, however, discuss one of the most surprising links between Number Theory and Harmonic Analysis, namely, the problem of characterizing sets of uniqueness among Cantor sets. We shall come back to this topic later in this review.

Let us first describe Pisot numbers, also known as PV-numbers (P for Pisot and V for Vijayaraghavan, who independently completed some important work on the topic) or as S-numbers, as Pisot himself coined them. (Contrary to what many mathematicians think, S is not in honor of Salem). This is somewhat confusing, since Salem numbers (discussed by Salem, of course) form yet another set named T!

Pisot numbers are real algebraic integers $\theta > 1$, all the conjugates of which if different from θ (if any) lie *in* the unit disc. In other words, if θ is of degree s and if $\theta^{(1)} = \theta, \theta^{(2)}, \theta^{(3)}, \dots, \theta^{(s)}$ are the conjugates, then

$$\theta > 1 \qquad \text{and} \qquad |\theta^{(k)}| < 1 \quad \text{for } k = 2, 3, \dots, s.$$

Examples of such numbers are all the rational integers greater than or equal to 2, the golden mean $(1+\sqrt{5})/2$, the real zero of the polynomial $x^3 - x - 1$, etc. Every real algebraic extension of finite degree s over $\mathbb{Q}$ contains infinitely many Pisot numbers of degree s.

Consider $(1+\sqrt{5})/2$ to a large power, say, 100:

$$\left(\frac{1+\sqrt{5}}{2}\right)^{100} = 792\,070\,839\,848\,372\,253\,126.999\,999\,999\,999\,999\,999\,998\,737\dots.$$

At the right of the decimal point there appears a string of twenty 9's which illustrates the fact that $((1+\sqrt{5})/2)^{100}$ is very nearly an integer. This is, of course, a rather trivial property of Pisot numbers. Indeed if θ is a Pisot number and if λ is a real algebraic integer in the field $\mathbb{Q}(\theta)$, then

$$\lambda\theta^n + \sum_{k=2}^{s} \lambda^{(k)}(\theta^{(k)})^n \in \mathbb{Z}$$

where $\lambda^{(k)}$ and $\theta^{(k)}$ denote the conjugates of λ and θ respectfully. For $k = 2, 3, \dots, s$ we know that $|\theta^{(k)}| < 1$ so that

$$\lim_{n\to\infty} \lambda\theta^n \equiv 0 \pmod 1.$$

The following question arises. Let $\lambda \neq 0$ and $\theta > 1$ be two real given numbers such that

$$\lim_{n\to\infty} \lambda\theta^n \equiv 0 \pmod 1.$$

Is it true that (θ, λ) is a "Pisot pair", i.e., θ is a Pisot number and $\lambda \in \mathbb{Q}(\theta)$? The question turns out to be extremely deep: to this day the problem is yet unsolved even though it dates from the beginning of the century (or even before?). In his 1938 thesis [4] Pisot found a remarkable result which gives a partial answer.

Define the "norm" on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by

$$\|x\| = \min_{n\in\mathbb{Z}} |x - n|.$$

He proved that, if $\lambda\theta^n$ tends to 0 (mod 1) sufficiently rapidly, then (θ, λ) is a Pisot pair. "Sufficiently rapidly" means

$$\sum_{n=0}^{\infty} \|\lambda\theta^n\|^2 < \infty$$

or equivalently

$$\prod_{n=0}^{\infty} |\cos \pi\lambda\theta^n| \text{ converges.}$$

These conditions can be relaxed slightly: the same conclusion holds if

$$\|\lambda\theta^n\| = \circ(1/\sqrt{n})$$

or even if for all large n

$$\|\lambda\theta^n\| < \frac{1}{5(\theta+1)^2}\frac{1}{\sqrt{n}}$$

(5 can be replaced by $2\sqrt{2} + \varepsilon$). See Theorems 5.4.2, 5.4.3, and 5.4.4.

Another result of Pisot's is the following. Suppose that $\theta > 1$ is a real algebraic number, and suppose there exists a real $\lambda \neq 0$ such that $\|\lambda\theta^n\| \to 0$ (mod 1). Here there is no condition on the speed with which the convergence takes place. Then again (θ, λ) is a Pisot pair. This leaves us with the irritating problem of deciding whether there exist transcendental numbers $\theta > 1$ and real nonzero λ's such that $\lambda\theta^n \to 0$ (mod 1). This is unanswered to this day. These questions are discussed in Chapter 5.

A trivial consequence of Pisot's second result is that, if p and q are coprime integers, $1 < q < p$, then the sequence $(p/q)^n$ does not converge to 0 (mod 1). (If it did, p/q would be a rational integer!) It is known that the sequence has infinitely many cluster points (Pisot, Vijayaraghavan, A. Weil). Is the sequence dense (mod 1) or even uniformly distributed? The answer is unknown. Choquet wrote a series of seven papers [1] devoted to the behavior of the sequence $\lambda(p/q)^n$, using ideas from the theory of Dynamical Systems, but could not solve the main problem.

Sequences (α^n) or, more generally, $(\lambda\alpha^n)$ where $\alpha > 1$ have played an essential role in the pioneering work of Pisot. It was known since Koksma (1935) that, for all real nonzero λ and for almost all $\alpha > 1$, the sequence $(\lambda\alpha^n)$ is uniformly distributed (mod 1). Pisot pairs (θ, λ) belong to the exceptional set. More generally, if θ is a Pisot number and if $\lambda \in \mathbb{Q}(\theta)$, then $(\lambda\theta^n)$ has a finite number of cluster points (mod 1), so it also belongs to the exceptional set.

It is very frustrating not to know any specific $\alpha > 1$ such that (α^n) is uniformly distributed (mod 1), even though almost all $\alpha > 1$ do share the property. We do, however, know real numbers $\tau > 1$ such that (τ^n) is dense (mod 1) but not uniformly distributed. This is where Salem numbers, otherwise known as T-numbers, come into the picture.

By definition a T-number is a real algebraic integer $\tau > 1$ such that all conjugates other then τ are *on* the unit disc with at least one conjugate with absolute value 1. It then can be shown that τ is necessarily of even degree at least 4. There is exactly one conjugate outside the unit disc (τ itself), there is exactly one conjugate *inside* the unit disc $\tau^{(1)}$, and all other conjugates lie on the circumference $|z| = 1$. If τ is a T-number, then the extension $\mathbb{Q}(\tau)$ is necessarily a real quadratic extension of a totally real field.

Let τ be a T-number of degree $2s + 2$. The conjugates are

$$\tau^{(0)} = \tau, \quad \tau^{(1)} = \tfrac{1}{\tau}, \quad \tau^{(j)} = e^{2i\pi\omega^{(j)}}, \quad 1/\tau^{(j)}, \qquad j = 2, 3, \dots, s.$$

Then clearly

$$\tau^n = \frac{1}{\tau^n} + 2\sum_{j=2}^{s} \cos 2\pi n\omega^{(j)}.$$

The frequencies $\omega^{(j)}$ together with 1 are shown to be $\mathbb{Z}$-linearly independent, from where it easily follows that τ^n is dense (mod 1) and not uniformly distributed.

Chapters 6 and 7 are devoted to the topological and metric properties of the sets S (set of Pisot numbers) and T (set of Salem numbers). In 1944 Salem disproved a conjecture of Pisot and Vijayaraghavan, much to the surprise of the mathematical community. He established that the set S is closed and therefore contains a smallest element which is shown to be the real root of the equation $x^3 - x - 1 = 0$, $\theta_0 = 1.3247, \dots$. Let S' be the derived set of S, i.e., the set of cluster points of S. S' contains all rational integers greater than or equal to 2, all totally real S-numbers, and all powers θ^m $(m \geq 2)$ of S-numbers,... . The least element of S' is the golden mean $(1 + \sqrt{5})/2$.

Let S'' be the derived set of S', and more generally let $S^{(k)}$ be the derived set of $S^{(k-1)}$ $(k = 2, 3, \dots)$. The authors of the book describe the structure of these sets and establish among other results that $\min S'' = 2$ and that

$$\sqrt{k} \leq \min S^{(k)} \leq k + 1.$$

In particular, none of the derived sets is empty.

What can be said about the set T of Salem numbers? The situation is more difficult to deal with, and it is still an open problem to decide whether or not there exists a smallest T-number. The conjecture, linked to a problem of Lehmer's, seems to be that Lehmer's tenth-degree polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

provides the smallest Salem number. As for the closure $\overline{T}$ of T, one proves that $\overline{T} \supsetneq S$.

We now consider once again "badly" distributed sequences (mod 1). Following Pisot, the authors consider pairs of real numbers (λ, α) with $\lambda > 0$ and $\alpha > 1$. They show that there exists only a countable set of pairs such that for all large n

$$\|\lambda\alpha^n\| < \frac{1}{2(1+\alpha)^2}.$$

The proof is remarkably simple and deserves to be reproduced. Define

$$u_n = \lambda\alpha^n + \|\lambda\alpha^n\| \in \mathbb{N}.$$

Then

$$|u_{n+2} - u_{n+1}^2/u_n| \leq \tfrac{1}{2}$$

for all large n so that

$$u_{n+2} = E(u_{n+1}^2/u_n), \qquad n \geq n_1,$$

where $E(x)$ is the nearest integer to x: $-\frac{1}{2} \leq x - E(x) < \frac{1}{2}$. The term u_{n+2} is thus determined from u_n and u_{n+1}, which implies that the sequence is completely determined once one knows u_{n_1} and u_{1+n_1}. There are countably many such sequences. Now observe that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \alpha \quad \text{and} \quad \lim_{n\to\infty} \frac{u_n^{n+1}}{u_{n+1}^n} = \lambda.$$

This establishes the claim.

This proof involves sequences defined by

$$u_{n+2} = E(u_{n+1}^2/u_n) \quad \text{for } n \geq 0.$$

These sequences, called Pisot sequences, play an important role and are studied *per se* in Chapter 13, which we now discuss.

Let $E(a_0, a_1)$ be the Pisot sequence such that $u_0 = a_0$, $u_1 = a_1$, $a_0, a_1 \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \gamma = \gamma(a_0, a_1) \begin{cases} = 1 & \text{if } a_0 \leq a_1 < a_0 + \sqrt{a_0/2}, \\ > 1 & \text{if } a_1 \geq a_0 + \sqrt{a_0/2}. \end{cases}$$

Furthermore, the set $E = \{\gamma(a_0, a_1)/a_0, a_1 \in \mathbb{N}\}$ is dense on $x \geq 1$. Also, $S \cup T \subset E$. If $\gamma \in S \cup T$, then the Pisot sequence $E(a_0, a_1)$ which corresponds to γ is recurrent; i.e., there exist rational integers $q_1 q_2 \cdots q_s$ such that

$$u_n + q_1 u_{n-1} + q_2 u_{n-2} + \cdots + q_s u_{n-s} = 0 \quad \text{for } n \geq n_0.$$

In 1977 Boyd established the existence of nonrecurrent Pisot sequences, e.g., $E(14, 23)$. The set of $\gamma(a_0, a_1)$ corresponding to nonrecurrent Pisot sequences is actually dense on the interval $[(1+\sqrt{5})/2, +\infty]$.

In order to have a better understanding of the set $S \cup T$, Boyd introduced new sequences à la Pisot which are discussed at length at the end of Chapter 13. We shall not describe them in this review for lack of space.

Maybe one of the most surprising facts linked to Pisot numbers is the Salem-Zygmund theorem characterizing sets of uniqueness of trigonometric series. Way back in the middle of the nineteenth century, Riemann proved that if the trigonometric series

$$\frac{a_0}{2} + \sum_{n\geq 1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx) = 0$$

for all $x \in (0, 1)$, then $a_n = b_n = 0$ for all n. The problem then arose whether the same conclusion holds under the milder condition that the series vanish for all $x \in (0, 1)$ with the possible exception of those x in some set $E \subset (0, 1)$. Such an E is called a set of uniqueness. Riemann's result states that the empty set $\varnothing$ is a set of uniqueness. Cantor, followed by others, proved that countable sets are sets of uniqueness. On the other hand, a set of positive measure is certainly not a set of uniqueness. So the problem was left to decide whether sets of measure 0 had to be sets of uniqueness or not.

Let $\alpha > 2$, and consider the Cantor set E_α of dissection α, i.e., the set of x's

$$x = (\alpha - 1)\sum_{k=1}^{\infty} \varepsilon_k \alpha^{-k}, \qquad \varepsilon_k = 0, 1.$$

For $\alpha = 3$ one obtains the famous triadic Cantor set. Quite obviously, for all $\alpha > 2$, E_α has measure 0. The Salem-Zygmund theorem states that E_α is a set of uniqueness if and only if α is a Pisot number, $\alpha > 2$.

The book under review contains many topics that were not discussed here and that have never appeared in previous books. The Schur algorithm and its extensions, Smyth's theorem, the set S_q, rational approximations of algebraic numbers, the Jacobi-Perron algorithm, rational functions over rings of adeles, Pisot and Salem numbers in adeles, Pisot elements in a field of formal power series, etc., are among topics that the reviewer has not discussed due to lack of space.

The book is well written and well presented. It is informative and precise. My only criticism is that it does not contain any subject index and that the references appear at the end of each of the fifteen chapters rather than at the end of the book.

Here is a short list of some misprints:

page 59, line 2: Read Chamfy.
page 80: What is the pertinence of λ in Theorem 5.1.2?
page 98: I doubt that Korneyai referred to Pisot in 1919.
page 167: T. Kamae is a co-author in [3] and the pages should be 369–384.
page 290: How could one forget Salem in reference [7]. Change 1972 to 1963.

Salem died in 1963, and Pisot in 1984. The beauty of their discoveries, updated by the work of many younger mathematicians, including their students (four among the five authors), will certainly fascinate the reader. The book, dedicated to Charles Pisot, is also a tribute to Rafaël Salem.

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