BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 2, October 1993

0273-0979/93 $1.00 + $.25 per page

Teichmüller theory in Riemannian geometry, by Anthony J. Tromba. Birkhauser-Verlag, Basel, 1992, 222 pp., $29.00. ISBN 3-7643-2735-9

One of the most appealing features of Riemann surfaces is the variety of mathematical fields in which they arise as a basic object of study. Indeed, the point behind the Riemann-Klein-Weyl (1851–1913) introduction of Riemann surfaces is that they are, at once, both one-dimensional complex manifolds and algebraic curves, properly belonging to both the fields of algebraic geometry and complex differential geometry. Of course, even at those times in the early history of Riemann surfaces, one could study Riemann surfaces as two-dimensional real manifolds, as Gauss (1822) had already taken on the problem of taking a piece of a smooth oriented surface in Euclidean space and embedding it conformally (i.e., preserving angles and orientation) into the complex plane. A fourth perspective emerged from the uniformization theory (1882–1907) of Klein-Poincaré-Koebe, who showed that every Riemann surface, which by definition is a connected surface equipped with a complex analytic structure, also admits a homogeneous Riemannian metric. In the case where the surface is a closed surface of genus g, this means that neighborhoods on the surface are isometric to neighborhoods on the round sphere, the flat Euclidean plane, or the constant negative curvature hyperbolic plane for $g = 0, 1$, or > 1, respectively. Poincaré (1882) then began a study of Fuchsian groups (discrete subgroups of $\mathrm{PSL}(2, \mathbf{R})$) and the relationship between the group theory and the homogeneous geometry: a basic example in this subject is a representation of the fundamental group of a closed surface of genus $g \geq 2$ as a (torsion free) subgroup Γ of $\mathrm{PSL}(2, \mathbf{R})$, with the quotient $O(2) \setminus \mathrm{PSL}(2, \mathbf{R})/\Gamma$ being a Riemann surface equipped with a hyperbolic metric.

In studying Riemann surfaces, one is naturally led to a study of their moduli and to consideration of various spaces of moduli of Riemann surfaces. We will focus on Teichmüller's moduli space of surfaces. Of course, each field of mathematics and each perspective on Riemann surfaces offers its own definition of the Teichmüller space T_g of closed surfaces of genus g: we will follow Tromba and adopt the viewpoint of Riemannian geometry as follows. Let M be a smooth closed oriented surface of genus g, and let $\mathscr{C}$ denote the space of all compatible complex structures c on M. We observe that a diffeomorphism $f: M \to M$ pulls back a complex structure c on M to a new complex structure f^*c on M. Thus the group Diff_o of diffeomorphisms homotopic to the identity acts on $\mathscr{C}$ by pullback, and we define the Teichmüller space $\mathscr{T}_g$ to be the quotient space $\mathscr{T}_g = \mathscr{C}/\mathrm{Diff}_o$.

The case of the torus $(g = 1)$ is understood completely and serves as a motivating example. Pick a pair of oriented curves (γ_1, γ_2) on $M = T^2$ that generate $\pi_1 M$ (and whose ordering is compatible with the orientation of M) for us to use to track the homotopy class of a diffeomorphism $f: M \to M$. Then, the uniformization theorem (along with some normalization) permits us to find, for each complex structure c on M, a conformal realization of the universal cover $\widetilde{M}$ of M as the complex plane $\mathbf{C}$ (so that the projection map

$p : \widetilde{M} = \mathbf{C} \to M = T^2$ is conformal) and the curves γ_1 and γ_2 correspond to the deck transformations $z \mapsto z + 1$ and $z \mapsto z + \tau$, where the choice of orientation forces Im $\tau > 0$. (Even here there is a choice of perspective in that we might have said that the uniformization theorem provides M with a flat metric $g(c)$ which lifts to a flat metric $\widetilde{g}(c)$ on $\widetilde{M}$; we then develop $\widetilde{M}$ onto the flat plane $\mathbf{C}$ with the deck transformations corresponding to γ_1 and γ_2 being carried over by the holonomy map to the Euclidean isometries $z \mapsto z + 1$ and $z \mapsto z + \tau$.) Thus, to a complex structure c we associate the complex parameter $\tau = \tau(c)$ in the upper half space $U = \{\tau \mid \operatorname{Im} \tau > 0\}$. Of course, since any quotient $\mathbf{C}/\langle z \mapsto z + 1, z \mapsto z + \tau\rangle$ is a torus, all parameter points τ in U are realizable, and, in fact, the parameter space U is precisely $\mathscr{T}_1$: If there were a $c \in \mathscr{C}$ and an $f \in \mathrm{Diff}_o$ s.t. $\tau(f^*c) = \tau(c)$, then the map f would lift to a holomorphic homeomorphism $F : \mathbf{C} \to \mathbf{C}$ (so that $F(z) = az + b$) whose composition with the generators $z \mapsto z + 1$ and $z \mapsto z + \tau$ of the deck transformation group were those same generators, i.e., $F(z) = z$, forcing $f =$ identity map.

Thus, we see that we have an explicit description $\mathscr{T}_1 \simeq U$ of $\mathscr{T}_1$ as a complex manifold which admits a complete Kahler metric (the Poincaré hyperbolic metric of negative sectional curvature) and which also possesses a rich function theory intimately connected with the metric and the metrically natural compactification $\overline{U} = U \cup \mathbf{R} \cup \infty$. Even without this inspiring example, the definition of $\mathscr{T}_g$ provokes many questions. Is $\mathscr{T}_g$ a manifold? Is it a complex manifold? How should we model the tangent and cotangent spaces? Are there natural metrics? Are there natural embeddings into $\mathbf{C}^n$ or $\mathbf{R}^{2n}$? Are there natural compactifications? What can be said about the function theory? The history of Teichmüller theory is a story of new perspectives on Riemann surfaces and maps between Riemann surfaces contributing increasingly sophisticated answers to questions further and further down this list.

Fricke and Klein ([FK]; see also [K, Ab]) thought of a Riemann surface in terms of the description of the uniformized Riemann surface as $O(2) \backslash \mathrm{SL}(2, \mathbf{R})/\Gamma$, where Γ is the image of a representation in $\mathrm{SL}(2, \mathbf{R})$ of the fundamental group $\pi_1 M$ of the surface. Here, we are to think of a point in $\mathscr{T}_g$ as being represented by the discrete subgroup $\Gamma \cong \pi_1 M$ (with two isomorphic subgroups representing the same point in $\mathscr{T}_g$ if and only if they are conjugate within $\mathrm{SL}(2, \mathbf{R})$). The group Γ can be completely determined in terms of some algebraic invariants (e.g., the traces) of a finite generating set of Γ; moreover, one can make a judicious choice of generators for $\pi_1 M$ so that the algebraic invariants, viewed as functions of Γ in $\mathscr{T}_g$, provide global real analytic coordinates on $\mathscr{T}_g$. The answer to the first question on our list as to the manifold structure on Teichmüller space is that $\mathscr{T}_g$ is topologically a $6g-6$-dimensional ball. (The same result was later obtained by Fenchel and Nielsen (see [Ab]) via a different, more geometric, view of the group Γ and the quotient $\mathbf{H}^2/\Gamma$, where Γ acts upon the hyperbolic plane $\mathbf{H}^2$ by isometries. They obtained geometric global coordinates for T_g, including, for example, the lengths of the geodesic representatives of a particular family of elements of $\pi_1 M$.)

The next several breakthroughs in this selective history are due to the complex function theorist Teichmüller ([T1, T2]; see also [Ab]), who, following Grötzsch, had the insight to focus on maps between inequivalent Riemann surfaces that were not conformal. After all, if two complex structures c_1 and c_2 on M are equivalent in $\mathscr{T}_g$ if and only if there is a holomorphic self-map $f:(M, c_1) \to (M, c_2)$ homotopic to the identity, then two complex structures c_1 and c_2 are inequivalent if and only if no such map is possible. Teichmüller also introduced variational problems to the subject by posing and solving the following variational problem: Given two inequivalent Riemann surfaces (M, c_1) and (M, c_2), find a map $f: M \to M$ in the homotopy class of the identity map $\mathrm{id}: M \to M$ which has the least (in the L^∞ sense) possible quasiconformal dilatation $K(c_1, c_2)$, which for a smooth f refers to the ratio of maximum to minimum eigenvalues of the differential df.

The (unique) solution to Teichmüller's problem depends on a space of tensors which are ubiquitous in Teichmüller theory, the space $A_2(c)$ of holomorphic quadratic differentials on a Riemann surface (M, c). These locally have the form $\phi(z)\,dz^2$ where $\phi(z)$ is holomorphic, and they admit a picture as an orthogonal pair of singular foliations on (M, c) (given locally by the lines $\{\operatorname{Re}\zeta = \text{const}\}$ and $\{\operatorname{Im}\zeta = \text{const}\}$ for the canonical coordinate $\zeta = \int^z \sqrt{\phi(z)\,dz^2}$ away from zeros of $\phi(z)$). The Teichmüller map has the form of a stretching by a factor e^t in the direction of one of these foliations and a shrinking by a factor e^{-t} in the other so that $t = 0$ corresponds to a conformal map, and, as $t \to \infty$, the resulting structures leave all compacta in $\mathscr{T}_g$. Teichmüller defines a complete metric d_T on $\mathscr{T}_g$ by $d_T(c_1, c_2) = \frac{1}{2}\log K(c_1, c_2)$. For a fixed holomorphic quadratic differential defining a fixed pair of singular foliations, the family of structures $\{c_t\}$ defined by letting t range along the real line determines a geodesic in this metric d_T with $d_T(c_0, c_t) = t$. This represents substantial progress on the third and fourth questions on our list: we now have the global structures on $\mathscr{T}_g$ of a metric and rays of geodesics, and we understand how these structures on $\mathscr{T}_g$ relate to geometric deformations of the surfaces which represent the points in $\mathscr{T}_g$.

The modern theory of Teichmüller spaces began in the 1950s and 1960s with Ahlfors and Bers, who founded a theory of Teichmüller spaces, Riemann surfaces, and Fuchsian groups upon the elliptic partial differential equation $w_{\bar z} = \mu w_z$ associated to a quasiconformal map. Here, important progress resulted from the broadening of their focus away from only the Teichmüller extremal quasiconformal maps to the space of all quasiconformal maps and the space of tensors $\mathscr{B} = \{\mu = w_{\bar z}/w_z\}$, called the space of Beltrami differentials, a space holomorphically equivalent to the space $\mathscr{C}$ of all complex structures used above. Ahlfors [A1] showed that Teichmüller space has a natural complex structure, and then Bers [A1] showed that the natural complex structure on $\mathscr{B}$ descended to $\mathscr{T}_g$, so that $\mathscr{T}_g$ was a complex manifold, with (Ahlfors [A2]) cotangent space $T_c^*\mathscr{T}_g = A_2(c)$ and tangent space naturally represented by a subspace $\mathscr{HB} \subset \mathscr{B}$ of Beltrami differentials that were harmonic with respect to the non-Euclidean metric on (M, c). Following a suggestion of Weil, Ahlfors [A2, A3] introduced a new metric on $\mathscr{T}_g$, the Weil-Petersson metric, which he showed to be Kähler and of negative holomorphic sectional curvature. Using quasiconformal deformations of Fuchsian groups, Bers [B] exhibited a holomorphic embedding of $\mathscr{T}_g$ into $\mathbf{C}^{3g-3}$, and later Bers and Ehrenpreis [BE] showed that $\mathscr{T}_g$ was a Stein manifold.

In the late 1970s Thurston [Th1] reintroduced purely geometric methods into Teichmüller theory, both for the development of the theory and also for the study of geometric structures on 3-manifolds. Central to his investigations was the hyperbolic metric structure on the Riemann surface, with its global structure and nontrivial deformations being encoded in the structure of the space of simple geodesics (both closed and not closed) on the hyperbolic surface. To get some idea of the change in viewpoint, consider a tangent vector to Teichmüller space, which in the last paragraph was the Weil-Petersson dual to a holomorphic quadratic differential. Thurston [Th2] explains that one deforms a hyperbolic surface by cutting the surface along a simple geodesic of a particular type (which are typically not closed—this introduces some subtleties into the operation), sliding one side of the cut to the right a small amount relative to the other side, and regluing. There are, of course, natural geometric and analytic correspondences [Wlp, HM] between the two viewpoints. In terms of our questions about Teichmüller space, which perhaps are somewhat irrelevant to this chapter, Thurston introduced a natural compactification for $\mathscr{T}_g$, acted upon naturally by the homotopy classes of diffeomorphisms (the mapping class group), which had the application of giving a very pretty classification of the diffeomorphisms of a surface.

More recently, other fields have started applying Teichmüller theory to their subjects. In dynamics Sullivan [S] has pioneered an approach to conformal dynamical systems using the Ahlfors-Bers quasiconformal theory, and there are now [KMS, V] clearly established connections between the dynamics of a billiard ball on certain polygonal tables and the dynamics of Teichmüller's geodesics on Riemann's moduli space of curves $\mathscr{R}_g = \mathscr{C}/\mathrm{Diff}_+$, a quotient of $\mathscr{T}_g$. Most germane to the subject of Tromba's book, however, are the applications to and from Riemannian geometry and physics which emerged in the 1980s.

Tromba's early research in Teichmüller theory was motivated by his interest in the Plateau-Douglas problem for higher genus surfaces, i.e., the problem, given k curves in $\mathbf{E}^m$ and a natural number $g \geq 1$, of finding an area-minimizing surface of genus g spanning the curves, provided that the infimum of area of connected (or disconnected, resp.) surfaces of genus $p < g$ $(p \leq g$, resp.) was not less than the infimum for area of connected surfaces of genus g. An approach to this problem along the original lines of Douglas and Courant (see [TT]) leads one to want to study the Dirichlet energy integral $D(u)$ for all maps $u : (M, c) \to \mathbf{E}^m$ of surfaces, and for genus $g \geq 1$ or $k > 1$ it is important that $D(u)$ depends upon the complex structure c of the domain. One is naturally led to a study of $D(u)$ and its critical points, the harmonic maps, in their dependence upon the complex structure c and the point it represents in moduli space. At about the same time, string theorists were also studying functionals on the space of all maps of surfaces into a target manifold, which gave added motivation to the global analysts' interest of providing a treatment of Teichmüller theory from a purely differential geometric perspective.

This is the purpose of Tromba's book, and he does an excellent job of answering many of the questions about the basic structure of Teichmüller space on our original list in a self-contained way from the Riemannian geometry and elliptic PDE point of view.

In particular, he takes a Riemann surface to be a two manifold equipped with a complex structure or, via uniformization (which he proves), a hyperbolic metric. Then $\mathscr{T}_g$ is a quotient space of the space $\mathscr{C}$ of complex structures or the space $\mathscr{M}_{-1}$ of hyperbolic metrics; and the tangent space, complex structure, and metric on $\mathscr{T}_g$ are inherited from the same structures on $\mathscr{C}$ or $\mathscr{M}_{-1}$. To compare Riemann surfaces representing different points in Teichmüller space, Tromba uses harmonic maps $u_{\gamma,h}:(M,\gamma)\to(M,h)$ (whose relevant theory he carefully develops) rather than, say, the quasiconformal maps of the complex function theorists. A large part of his theory, and this book is basically a summary of some of the research of Tromba and coauthors on a Riemannian approach to Teichmüller theory, focuses on properties of the energy $E(\gamma,h)$ of the harmonic map γ, h between two surfaces, viewed as a function of (γ,h) in $\mathscr{T}_g\times\mathscr{T}_g$: parts of the basic structure of T_g are also consequences of properties of $E(\gamma,h)$.

The book is well written, with new topics and techniques being carefully motivated, even for the reader who is not an expert in either Riemannian geometry or Teichmüller theory. Of course, the discussion becomes technical in places, but the well-organized exposition allows the reader to skip the more technical passages on the first reading; the expert may wish to follow those passages carefully, as they contain genuinely new material and are not merely a translation of older proofs into new language.

The only area in which the exposition falters is in the author's description of the book's scope as an introduction. The book covers many foundational results in Teichmüller theory from a single perspective; however, there are many foundational results that are omitted—for instance, the theory of Teichmüller maps and the Teichmüller metric, the Bers global holomorphic embedding of $\mathscr{T}_g$ into $\mathbf{C}^{3g-3}$, and the Thurston theory. Some of this material, like the Bers embedding, is not currently accessible via the author's approach, whereas some of it, like a portion of the Thurston theory [W, M], is accessible. Further bibliographical remarks would have greatly improved this introduction. Still, Tromba's monograph covers much ground in a self-contained manner accessible to a wide audience distinct from that catered to by other recent books on Teichmüller theory and so represents an important addition to the literature of this multifaceted field.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 2, October 1993

0273-0979/93 $1.00 + $.25 per page

Manifolds with singularities and the Adams-Novikov spectral sequence, by Boris I. Botvinnik. Cambridge University Press, London and New York, 1992, xv+181 pp., $37.95. ISBN 0-521-46208-1

From the preface: "The purpose of this book is to discuss some natural relations between geometric concepts of Cobordism Theory of manifolds with singularities and the Adams-Novikov spectral sequence." Before discussing the book itself, we will give some background and define the terms used in its opening sentence.

Cobordism theory of manifolds (without singularities) was one of the great successes of algebraic topology in the 1950s and 1960s. The basic idea is the following: Two closed (i.e., compact, smooth, and without boundary) n-dimensional manifolds, M_1 and M_2, are said to be *cobordant* if there is a smooth $(n+1)$-dimensional manifold W whose boundary is the disjoint union of M_1 and M_2.

In particular, M_2 could be empty, in which case we are requiring M_1 to be the boundary of some W. The most easily visualized closed manifolds, namely,