RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 2, October 1993

COUPLING AND HARNACK INEQUALITIES FOR SIERPINSKI CARPETS

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ABSTRACT. Uniform Harnack inequalities for harmonic functions on the pre- and graphical Sierpinski carpets are proved using a probabilistic coupling argument. Various results follow from this, including the construction of Brownian motion on Sierpinski carpets embedded in $\mathbb{R}^d$, $d \geq 3$, estimates on the fundamental solution of the heat equation, and Sobolev and Poincaré inequalities.

The Sierpinski carpets (SCs) we will study are generalizations of the Cantor set. Let $F_0 = [0, 1]^d$ be the unit cube in $\mathbb{R}^d$, $d \geq 2$, centered at $\mathbf{z}_0 = (1/2, \dots, 1/2)$. Let k, a be integers with $1 \leq a < k$ and $a + k$ even. Divide F_0 into k^d equal subcubes, remove a central block of a^d subcubes, and let F_1 be what remains: thus $F_1 = F_0 - ((k-a)/2k, (k+a)/2k)^d$. Now repeat this operation on each of the $k^d - a^d$ remaining subcubes to obtain F_2. Iterating, we obtain a decreasing sequence of closed sets F_n; then $F = \bigcap_{n=0}^{\infty} F_n$ is a Sierpinski carpet and has Hausdorff dimension $d_f = d_f(F) = \log(k^d - a^d)/\log(k)$. (When $d = 2$, $k = 3$, and $a = 1$, we get the usual Sierpinski carpet.) Let $\widehat{F}_n = k^n F_n \subset [0, \infty)^d$, and define the *pre-Sierpinski carpet* by $\widehat{F} = \bigcup_{n=1}^{\infty} \widehat{F}_n$ (see [10]). The *graphical Sierpinski carpet* is the graph $G = (V, E)$ with vertex set $V = (\mathbf{z}_0 + \mathbb{Z}^d) \cap \widehat{F}$ and edge set $E = \{\{x, y\} \in V : |x - y| = 1\}$.

Thus $\operatorname{int}(\widehat{F})$ is a domain in $\mathbb{R}^d$ with a large-scale structure which mimics the small-scale structure of F. We are interested in the behavior of solutions of the Laplace and heat equations on F, $\widehat{F}$, and G. One reason for this is applications to "transport phenomena" in disordered media (see [6]); another is the new type of behavior of the heat kernel on these spaces. Let W be Brownian motion on $\widehat{F}$ with normal reflection on $\partial\widehat{F}$, and let $q(t, x, y)$ be the transition density of W, so that q solves the heat equation on $\widehat{F}$ with Neumann boundary conditions on $\partial\widehat{F}$.

Received by the editors November 9, 1992.
1991 *Mathematics Subject Classification*. Primary 60B99; Secondary 60J35.
Key words and phrases. Harnack inequality, Sierpinski carpets, self–similar, fractals, Brownian motion, heat equation, transition densities, Poincaré inequality, Sobolev inequality, spectral dimension, electrical resistance.
Research partially supported by NSF grant DMS 91–00244.

Theorem 1. *There exist* $c_1, \ldots, c_6 \in (0, \infty)$ *and* $d_s = d_s(F) \in (1, d_f)$ *such that if* $x, y \in \widehat{F}$, $t \in (1, \infty)$, $|x-y| \le t$, *then*

$$\begin{aligned} &c_1 t^{-d_s/2} \exp\left(- c_2 \left(\frac{|x-y|^{d_w}}{t} \right)^{1/(d_w - 1)} \right) \\ &\qquad \le q(t, x, y) \le c_3 t^{-d_s/2} \exp\left(- c_4 \left(\frac{|x-y|^{d_w}}{t} \right)^{1/(d_w-1)} \right), \end{aligned} \tag{1}$$

where $d_w = 2d_f/d_s$; *while if* $x, y \in \widehat{F}$, $t \in (1, \infty)$, $|x-y| > t$, *then*

$$\exp\left(-c_5 \frac{|x-y|^2}{t}\right) \le q(t, x, y) \le \exp\left(-c_6 \frac{|x-y|^2}{t}\right). \tag{2}$$

The index d_s is called the *spectral dimension* of F and turns out to be much more significant than the Hausdorff dimension d_f as far as analytic properties of these spaces are concerned. Since $d_s < d$, (1) confirms the physical intuition that the presence of increasingly large reflecting barriers causes heat to dissipate to infinity more slowly. It seems unlikely that there is any simple relationship between d_s, k, a, and d.

While there is a well-developed approach to the heat equation using analytic tools such as Sobolev or log-Sobolev inequalities (see [7]), these methods do not appear to give the best-possible results on spaces such as $\widehat{F}$—compare the upper bound on $q(t, x, y)$ given in Theorem 1 with the results of [10].

The proof of Theorem 1 rests on the following Harnack inequality. Let $D \subset \mathbb{R}^d$ be open: we will say that h is *harmonic on* $D \cap \widehat{F}$ if (i) $\Delta h = 0$ in $\operatorname{int}(D \cap \widehat{F})$ and (ii) h has 0 normal derivative a.e. on $D \cap \partial \widehat{F}$. Equivalently, h is harmonic with respect to $W(t \wedge T_D)$, where $T_D = \inf\{t : W(t) \notin D\}$. Let $D_n = (-1, k^n)^d$.

Theorem 2. *There exists* $c_1 \in (0, \infty)$ (*depending only on* d, k, a), *such that if* h *is positive harmonic in* $D_n \cap \widehat{F}$ *and* $x, y \in D_{n-1} \cap \widehat{F}$, *then* $h(x)/h(y) \le c_1$.

Remarks. 1. Note that c_1 is independent of n; otherwise the result is trivial.

2. The case $d = 2$ was proved in [1]; the proof there relies on the fact that a closed curve in the plane separates the plane into two pieces. Just as in the case of elliptic operators, the results for two dimensions are considerably easier to prove. The result of [1] was extended in [8] to SCs with $d_s(F) < 2$.

3. Using the symmetry of $\widehat{F}$, Theorem 2 extends to other domains in $\widehat{F}$.

4. Theorems 1 and 2 actually hold for a much wider class of SCs, those satisfying a higher-dimensional generalization of (2.1) of [4].

5. A similar result holds for the graphical Sierpinski carpet G.

6. Most existing proofs of Harnack inequalities for selfadjoint operators depend on Sobolev inequalities, which in turn depend on the underlying geometry of the space. Here the appropriate Sobolev inequality involves the spectral dimension $d_s(F)$; however, no geometric definition of d_s is known. Thus we were led to abandon analytic approaches in favor of the probabilistic coupling argument described at the end of this paper.

We now describe some other consequences of Theorem 2. Let $\widetilde{F} = \bigcup_{n=0}^{\infty} k^n F$, the SC extended to $[0, \infty)^d$, and write μ for Hausdorff x^{d_f}-measure on $\widetilde{F}$.

Theorem 3. *There exists a strong Markov process X_t with state space $\widetilde{F}$ such that X has a strong Feller transition semigroup P_t which is μ-symmetric, X_t has continuous paths, X_t is self-similar with respect to dilations of size k^n, and the process X is locally invariant with respect to the local isometries of $\widetilde{F}$.*

Let $p(t, x, y)$ be the transition density of X_t with respect to μ. Then $p(t, x, y)$ is the fundamental solution to the heat equation on $\widetilde{F}$: $\partial u/\partial t = \Delta_{\widetilde{F}} u$, where $\Delta_{\widetilde{F}}$ is the infinitesimal generator of X_t. Then we have

Theorem 4. *There exist $c_1, c_2, c_3, c_4 \in (0, \infty)$ and $d_s = d_s(F) \in (1, d_f)$ such that for all $x, y \in \widetilde{F}$, $t \in (0, \infty)$,*

$$c_1 t^{-d_s/2} \exp\left(-c_2\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \le p(t, x, y) \le c_3 t^{-d_s/2} \exp\left(-c_4\left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right),$$

where $d_w = 2d_f/d_s$. Moreover, $p(t, x, y)$ is C^∞ in t, and $p(t, x, y)$ and all its partial derivatives with respect to t are jointly Hölder continuous in x and y.

Many properties of the process X, such as its transience or recurrence, the existence of local times, the existence of self-intersections, and the asymptotic frequency of eigenvalues follow easily from Theorem 4. For example, note that X is point recurrent if and only if $d_s(F) < 2$.

The next set of consequences include Sobolev inequalities, Poincaré inequalities, and electrical resistance inequalities for $\widetilde{F}$, $\widehat{F}$, and G—nine theorems in total. Since the electrical resistance inequalities are probably the least well-known type, we give the one for G as a representative sample. If B is any subset of G, let $|B|$ denote the cardinality of B. Then $R(B)$, the resistance from B to infinity, is defined by

$$R(B)^{-1} = \inf\left\{ \sum_{\{x,y\}\in E(G)} (f(x) - f(y))^2 : f \equiv 1 \text{ on } B,\ f(x) \to 0 \text{ as } |x| \to \infty \right\}.$$

The inverse of $R(B)$ is the conductance from B to infinity and equals the capacity of B.

Theorem 5. *Suppose $d_s = d_s(F) > 2$, and let $\zeta = d_s/(d_s - 2)$. Then there exists c_1 such that if $A \subset G$, $|A| \le c_1 R(A)^{-\zeta}$.*

Theorem 5 follows fairly straightforwardly from Theorem 4 by applying ideas of [11] and [12]. As Theorems 1, 3, and 4 follow from Theorem 2 by generalizations and modifications of methods of [1–4, 8, 9], we discuss only Theorem 2.

Let W_t be the Brownian motion on $\widehat{F}$ described above, and let

$$\tau(x, r) = \inf\{t : |W_t - x| \ge r\}, \qquad T(x, r) = \inf\{t : |W_t - x| \le r\}.$$

The following lemma is proved in a similar fashion to Lemma 3.2 of [1].

Lemma 6. *There exist* $c_2 > c_1 > 1$, $\delta > 0$ *independent of* r *such that if* $x, y \in \widehat{F}$ *and* $|y - x| \le c_1 r$, *then*

$$\mathbb{P}^y(T(x, r) < \tau(x, c_2 r)) > \delta. \tag{3}$$

It is known (see Theorem 3.9 of [5], for example) that the Harnack inequality Theorem 2 follows from (3) and an oscillation inequality of the following form.

Lemma 7. *There exists* $\rho < 1$ *such that if* $n \ge 1$ *and* h *is positive harmonic on* $D_n \cap \widehat{F}$, *then*

$$|h(x) - h(y)| \le \rho \sup_{z \in D_n \cap \widehat{F}} |h(z)|, \qquad x, y \in D_{n-1} \cap \widehat{F}. \tag{4}$$

To show (4), it suffices to construct two $\widehat{F}$-valued Brownian motions W^x and W^y, starting from x and y respectively, which couple (i.e., meet) with probability at least $1 - \rho$ before either exits D_n.

Fix n. Let $\mathcal{S}_m$ be the collection of cubes of side length k^m with vertices in $k^m \mathbb{Z}^d$. Say that $x, y \in \widehat{F}$ are m-associated if there is an isometry of the cube in $\mathcal{S}_m$ containing x onto the cube in $\mathcal{S}_m$ containing y that maps x onto y. Note that if two points are m-associated, then they will also be ℓ-associated for all $\ell \le m$.

Suppose first that x and y are m-associated. We start a Brownian motion $W^x(t)$ on $\widehat{F}$ at x. Let $U_0 = 0$, and $U_{i+1} = \inf\{t : |W^x(t) - W^x(U_i)| \ge k^m\}$. The key step is to exploit the local symmetry of $\widehat{F}$ to construct, using suitable reflections, another Brownian motion $W^y(t)$ on $\widehat{F}$, starting at y, such that (a) $W^x(t)$ and $W^y(t)$ are m-associated for all $t \ge 0$, and (b) there exist j and $c_1 > 0$ such that

$$\mathbb{P}\Big(W^x(U_j(\omega)) \text{ and } W^y(U_j(\omega)) \text{ are } (m+1)\text{-associated}\Big) > c_1. \tag{5}$$

A renewal argument and then an induction show that if x and y are 0-associated, then W^x and W^y couple with probability $c_2 > 0$ before either process leaves D_n. Lemma 6 then follows easily.

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