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WILLIAM M. BOOTHBY
WASHINGTON UNIVERSITY

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Systems of evolution equations with periodic and quasiperiodic coefficients, by Yu. A. Mitropolsky, A. M. Samoilenko, and D. Martinyuk. Mathematics and Its Applications (red series), Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993, xiv+280 pp., $126.00. ISBN 0-7923-2054-9

To unify somehow the content of the book, the authors have chosen the key words "Evolution Equations". It is not my intent to propose another key word, but I have the feeling that the real unifying element is the flavor of the Kiev School on Nonlinear Oscillations, initiated about six decades ago by Krylov and Bogoliubov. Let us mention some of the main publications that are characteristic of the development of this school. It started with Nicolai Mitrofanovich Krylov (born in 1879 and educated in St. Petersburg but working in Kiev since 1922). If one looks at the list of papers published by Krylov and by Bogoliubov and Krylov, one finds some of the same themes that are present in the book under review, such as "Approximation of periodic solutions of differential equations", a paper in French published in 1929 by Krylov, or "On the quasiperiodic solutions of the equations of the nonlinear mechanics", published in French in 1934 by Krylov and Bogoliubov.

The most famous publication of Krylov and Bogoliubov is their book *Introduction to nonlinear mechanics*, published in Kiev in 1937. What was typical of all these pioneering works of the Kiev School was the stress on computational aspects, on motivations, and on applications. In the sequel the main theoretical aspects were brilliantly clarified by N. N. Bogoliubov in a rather unknown monograph entitled *On certain statistical methods in mathematical physics* (Kiev) published in 1945; it was there the "method of averaging" found a full mathematical justification and the idea of reducing the problem by considering integral manifolds was pointed out.

The next event was the widely known book by Bogoliubov and Mitropolskii, *Asymptotic methods in the theory of nonlinear oscillations*, published in Russian in 1955, followed by a long series of books by Mitropolskii and his colleagues; among them let us mention that by Mitropolskii and Lykova, *Integral manifolds in nonlinear mechanics* (Nauka, Moscow, 1974). After discovering that the main ideas of the "accelerated convergence" procedure of Kolmogorov, Arnold, and Moser were already present in the work of Bogoliubov in the 1969 monograph *The methods of rapid convergence in nonlinear mechanics* (Naukova Dumka, Kiev), Yu. A. Mitropolskii and A. M. Samoilenko gave the Kiev version of this approach. Under the direction of Mitropolskii in Kiev a large number of studies were performed, dedicated mostly to the extension of the methods to different classes of equations and to various applications.

Now, what is the place of the book under review in this general picture? The book is constituted by putting together papers of the authors and their coworkers on the application of the ideas of the Kiev School to differential-difference equations and to difference equations (discrete-time systems). We find here the themes of approximation of periodic solutions, of quasiperiodicity, and of integral manifolds and averaging. The one theme of the Kiev School that is totally missing is the stress on applications; in fact, in the whole book there is not a single application.

The problem of periodic solutions of nonlinear equations appeared a long time ago, mostly in connection with celestial mechanics, and the attention focused on approximate solutions.

It was in the middle of this century that a switch of pure existence results by using topological methods was observed. As a reaction to this direction an important approach was proposed by Minoru Urabe in his paper "Galerkin procedure for nonlinear periodic systems", Arch. Rational Mech. Anal. **20** (1965), 120–152. I remember a discussion with Urabe: I was defending the existence proofs based on topological methods, and he replied, "I don't like statements such as 'there is a fish in the ocean'; I am interested in catching it." After more than a quarter of a century, I believe that Urabe was right.

Chapters 2 and 3 of the book under review use the scheme proposed by Urabe for the case of delay equations and for some partial differential equations for which "multiple periodic" solutions lead to quasiperiodic solutions for delay equations. The same theme of approximation of periodic solutions is present in the first chapter, again for delay-systems mainly, but also for other classes such as integro-differential equations, operator-differential equations, and difference equations. In fact, this is a rather simple idea of approximating by trigonometric polynomials which works under very severe assumptions, and the only interesting case where it applies is in furnishing a new proof of the main result of Bogoliubov concerning existence of a periodic solution for systems in "standard form" (slow motions) corresponding to an equilibrium of the "averaged" system. To prove the result of Bogoliubov, the ideas of Urabe could also be used.

As already mentioned one of the major themes of the Kiev School was that of invariant manifolds, motivated by the fact that in many situations individual periodic or quasiperiodic solutions may be unstable, while invariant manifolds containing them are stable. By using the usual Kiev School ideas, the book under review gives results concerning such invariant manifolds for the cases of delay equations and difference equations.

It is to be regretted, however, that the important contributions of Jaroslav Kurzweil, whose name is given in the preface as one in a list, are not mentioned in the references; in my opinion Kurzweil's approach is the most fundamental for the subject of invariant manifolds and clarifies it in the general setting of abstract evolutionary processes.

The method of accelerated convergence is used in a short chapter of the book to give a solution to the following problem: consider the system $x_{n+1} = Ax_n + P(\varphi_n)x_n$, $\varphi_{n+1} - \varphi_n = \omega$, where $x_n \in \mathbb{R}^s$, $\varphi_n \in \mathbb{R}^m$, P is periodic in each of $\varphi^1, \ldots, \varphi^m$ with period 2π, and $\omega = (\omega^1, \ldots, \omega^m)$ is constant; find a change of variables $x_n = \Phi(\varphi_n)y_n$, with periodic Φ, such that the above system is reduced to $y_{n+1} = Ay_n$, $\varphi_{n+1} - \varphi_n = \omega$.

What can one say after looking at the book? The Kiev School has known its time of flourishing with deep results and interesting applications. The book under review corresponds to the period when the only new stuff is merely the consideration of old ideas in some new area.

Let us end by mentioning the many places where the translation is unsatisfactory, which unfortunately is not the first such occurrence for translations from Russian made by nonspecialists. I want only to mention that many names of authors that were written with a transcription in Cyrillic were reconverted in Latin incorrectly. Here we note: Pew for Pugh, Peysoto for Peixoto, Yu. Mozer for J. Moser, Ya. Kurzweil for J. Kurzweil, Bol for Bohl, and Veksler for Wexler. Fortunately, the number of references to non-Russian-speaking authors is not very large.

ARISTIDE HALANAY
BUCHAREST UNIVERSITY

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Multiplication of distributions and applications to partial differential equations, by M. Oberguggenberger, Pitman Research Notes in Mathematics Series, vol. 259, Longman Scientific & Technical, Harlow, 1992, xvii+312 pp., \$39.00. ISBN 0-582-08733-3

Multiplication of distributions and, in fact, various *nonlinear* operations on distributions or generalized functions happen to have quite a history. Some of the early sources of interest were in physics, where the Dirac delta-function δ has brought with it both convenient calculation methods and puzzlement about their rigorous justification. For instance, in quantum mechanics one would like to play with formulas such as

$$\begin{aligned} \delta \cdot (1/x) &= -\delta'/2, \\ \delta^2 - (1/x)^2/\pi^2 &= -(1/x^2)\pi^2, \\ (\delta_+)^2 &= -\delta'/4\pi i - (1/x^2)/4\pi^2, \\ (\delta_-)^2 &= \delta'/4\pi i - (1/x^2)/4\pi^2, \end{aligned}$$

where $\delta_+ = (\delta + (1/x)/\pi i)/2$ and $\delta_- = (\delta - (1/x)/\pi i)/2$; see [G-DS, M] or pages 18–20 of the book under review. However, such formulas could hardly be justified, except for certain rather ad hoc and questionable computational manipulations.

Another source of interest came from *nonlinear shock waves*. Indeed, even in the case of the basic equation

$$U_t(t, x) + U(t, x)U_x(t, x) = 0, \qquad t \geq 0, \ x \in \mathbb{R},$$