BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 2, April 1994

0273-0979/94 $1.00 + $.25 per page

Fourier integrals in classical analysis, by Christopher Sogge. Cambridge University Press, Cambridge, 1993, x+237 pp., $39.95. ISBN 0-521-43464-5

Recall the Euclidian Fourier transform,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \tag{1}$$

and the Fourier inversion formula,

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi. \tag{2}$$

Initially, $\hat{f}$ is defined and (2) is established for nice functions, say, $f \in C_0^\infty(\mathbb{R}^n)$; but in asking in what sense (2) holds for general $f \in L^p(\mathbb{R}^n)$, one is led to an interesting circle of problems. A central place is occupied by the *Bochner-Riesz means*, defined for $0 < R < \infty$ and $\alpha > 0$ by

$$S_R^\alpha f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\alpha \hat{f}(\xi)\, d\xi, \tag{3}$$

which are the higher-dimensional analogues of the familiar Cesàro means from Fourier series of one variable. The key question is: for which α are the operators $\{S_R^\alpha\}_{0<R<\infty}$ uniformly bounded on $L^p(\mathbb{R}^n)$? (This allows one to interpret (2) as: $f = \lim_{R\to+\infty} S_R^\alpha f$, with the limit being in the L^p norm.) This question has been fully answered only in two dimensions and has been an important stimulus to research in Fourier analysis for many years.

Related to (3) is the restriction operator, R, which gives the restriction of the Fourier transform $\hat{f}(\xi)$ to the unit sphere $S^{n-1} = \{\xi : |\xi| = 1\}$, and its adjoint,

$$R^* g(x) = \int_{S^{n-1}} e^{ix\cdot\xi} g(\xi)\, d\sigma(\xi), \tag{4}$$

where $d\sigma$ is the usual surface measure on the sphere. Naively, Rf should not even be defined for $f \in L^p$, $1 < p \le 2$, since $\hat{f} \in L^{p'}$, $1/p + 1/p' = 1$, by the Hausdorff-Young inequality, and an $L^{p'}$ function is only defined almost everywhere, but in fact the Tomas-Stein restriction theorem holds:

$$\|Rf\|_{L^2(S^{n-1})} \le C\|f\|_{L^p(\mathbb{R}^n)}, \qquad 1 \le p \le \tfrac{2(n+1)}{n+3}, \tag{5}$$

and thus

$$\|R^* g\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^2(S^{n-1})}, \qquad \tfrac{2(n+1)}{n-1} \le q \le \infty. \tag{6}$$

These and related results have applications to the boundedness of the Bochner-Riesz means, estimates for solutions of the wave and Klein-Gordon equations, and unique continuation for solutions of elliptic partial differential equations [4, 14, 6, 7].

Recall also a fundamental result of real variable theory, the Lebesgue differentiation theorem: for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$(7)\quad f(x) = lim_{r\to 0}\frac{1}{|B(x, r)|}\int_{B(x,r)} f(y)\,dy, \qquad B(x, r) = \{y : |y-x| \le r\}.$$

Nowadays this is usually presented as a corollary of the Hardy-Littlewood maximal theorem,

$$(8)\quad \left\|\sup_{0<r<\infty}\frac{1}{|B(x, r)|}\int_{B(x,r)} |f(y)|\,dy\right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p \le \infty.$$

In the mid-seventies Stein [13] showed that for sufficiently large values of p, the set being integrated over can be changed from the solid ball $B(x, r)$ to the sphere $S(x, r) = \{y : |y-x| = r\}$:

$$(9)\quad \left\|\sup_{0<r<\infty}\frac{1}{|S(x, r)|}\int_{S(x,r)} |f(y)|\,d\sigma(y)\right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)},$$
$$n/n-1 < p \le \infty, \quad n \ge 3.$$

(Again, since $f \in L^p$ is not defined on sets of measure zero, it is not obvious that these integrals of $|f|$ are even defined.) The two-dimensional case of (9) remained open for a decade, being finally proved by Bourgain [1].

These results, their "variable coefficient" generalizations, and much more are the subject of *Fourier integrals in classical analysis.* The book focuses on estimates for oscillatory integrals, which capture the curvature of S^{n-1} which underlies (5) and (9). Augmented by rescaling and orthogonality arguments, these estimates prove to be powerful tools in establishing not just restriction and maximal theorems but also convergence results for eigenfunction expansions on general compact riemannian manifolds and space–time estimates for solutions of variable coefficient wave equations.

After two chapters of introductory material, the book's main themes are introduced in Chapter 2. Given a phase function $S(x, y)$ and an amplitude $a(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^m$, one is interested in the family of oscillatory integral operators

$$(10)\qquad T_\lambda f(x) = \int e^{i\lambda S(x,y)} a(x, y) f(y)\,dy, \qquad \lambda \in \mathbb{R}\backslash 0,$$

and in particular the decay rate (as $\lambda \to \infty$) of the $L^p(\mathbb{R}^m) \to L^q(\mathbb{R}^n)$ operator norm of T_λ. In the nondegenerate situation, where $n = m$ and $\det(d^2_{x,y}S) \ne 0$, one has the estimate $\|T_\lambda\|_{L^p \to L^{p'}} = O(\lambda^{-n/p'})$, $1 \le p \le 2$. The case $n = m+1$ relevant to the restriction theorem and Bochner-Riesz means is then dealt with assuming the *Carleson-Sjölin condition*, which imposes nondegeneracy on both the second and third derivatives of $S(x, y)$. This is used to prove the L^p boundedness of the Bochner-Riesz means in $\mathbb{R}^2$; the proof of the boundedness of the associated maximal operators [2] then leads to a (new) proof of Bourgain's theorem [9].

After a nice introduction to pseudodifferential operators in Chapter 3, a Weyl-type theorem on the asymptotic distribution of eigenvalues of first-order pseudodifferential operators and related results are proved in Chapter 4. The variable coefficient spectral theory is continued in the next chapter, which contains results on generalized Bochner-Riesz means and other functions of pseudodifferential operators on compact manifolds. All of this can be readily specialized to the most interesting case of functions of the Laplace-Beltrami operator on a riemannian manifold, but the more general formulations allow one to see what is really needed for the proofs.

In order to treat variable coefficient versions of the spherical and circular maximal operators, and even more general maximal operators, Fourier integral operators (FIOs) are introduced in Chapter 6. As the author states, this is not intended as a comprehensive guide to FIOs, and the interested reader can find the material presented from somewhat different points of view in [5] and [3].

Chapter 7 is the culmination of the book, where the recent results of Mockenhaupt, Seeger, and Sogge [10] on local smoothing for variable coefficient wave equations are proved. Consider the solution of the Cauchy problem for the wave equation in $\mathbb{R}^{n+1}$,

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\right) u(x, t) = 0, \qquad u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x). \tag{11}$$

It has been known for some time [11, 8] that there is a *fixed-time* estimate

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \le C(\|f\|_{L^p_\alpha} + \|g\|_{L^p_{\alpha-1}}), \qquad 1 < p < \infty, \tag{12}$$

where $\|f\|_{L^p_\alpha} = \|(I-\Delta)^{\alpha/2} f\|_{L^p(\mathbb{R}^n)}$ is the usual norm on the Sobolev space of distributions with α derivatives in L^p, and $\alpha = (n-1)|\frac{1}{p} - \frac{1}{2}|$. However, if one integrates over both space *and* time, a markedly improved estimate, referred to as *local smoothing*, is possible: if $p > 2$ and $K \subset\subset \mathbb{R}^{n+1}$, the left-hand side of (12) can be replaced by $\|u(\cdot, \cdot)\|_{L^p_\epsilon(K)}$ for some $\epsilon = \epsilon_p > 0$. Similar estimates were previously known for dispersive equations such as the Schrödinger equation; in the wave equation context they were introduced by Sogge [12]. This striking extra regularity can be applied to obtain variable coefficient improvements of the spherical and circular maximal theorems. Local smoothing holds for Fourier integral operators (including the solution operator $(f(\cdot), g(\cdot)) \to u(\cdot, \cdot)$ for (11)) which take functions of n variables to functions of $n+1$ variables and satisfy a *cinematic curvature* condition, which is an analogue of the Carleson-Sjölin condition. Cinematic curvature in two variables was recognized by Sogge as a property of circles crucial to Bourgain's proof of the circular maximal theorem. He then reformulated it in the language of Fourier integral operators and used it to prove variable coefficient versions of Bourgain's theorem. This condition is easily translated into higher dimensions, and the local smoothing for solutions to (11) follows from L^p estimates for general FIOs satisfying it [10]. All this, and more, is in Chapter 7.

Fourier integrals in classical analysis generally provides an excellent introduction to oscillatory integral operators and a detailed treatment of some of the most recent developments. Although based on notes of a graduate course, parts of it may be rough going for students: some of the most technical material, especially in §§2.4, 7.1, and 7.2, has not been changed much from the original research articles and would perhaps have benefited from a more relaxed pace, with more motivation of the numerous decompositions which are employed. The book is remarkably free of typos, but some pedagogic errors have crept in. On page 222, for example, the transverse intersection calculus for FIOs, which has been proved, is invoked, while what is actually needed is the clean intersection calculus, which has not.

But the above are quibbles; *Fourier integrals in classical analysis* rewards the reader with a thorough account of some of the last decade's most important developments in Fourier analysis, many of them due to its author. It belongs on the bookshelf of anyone seriously interested in the subject.

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AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 2, April 1994

0273-0979/94 $1.00 + $.25 per page

Introduction to the general theory of singular perturbations, by S. A. Lomov. Translations of Mathematical Monographs, vol. 112, American Mathematical Society, Providence, RI, 1992, xviii + 375 pp., $201.00. ISBN 0-8218-4569-1

The importance of the concept of "perturbation" has long been recognized in the field of celestial mechanics. In particular, the study of the motion of, say, a