BOOK REVIEWS

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Lectures on mechanics, by Jerrold E. Marsden. Cambridge University Press, Cambridge, 1992 (first ed. 1989), xi + 254 pp., $34.95. ISBN 0-521-42844-0

The study of mechanics, particularly the dynamics of planetary systems, was for several centuries at the core of mathematics. From Newton onwards through the further developments of Lagrange, Hamilton, and others an appropriate framework was created in which complex dynamical problems could be investigated. Because of its central importance, it fostered the early development of many concepts and techniques which now permeate the whole of mathematics, beginning of course with the calculus but including also key ideas in geometry.

The relevance of geometry to mechanics was clear from the start, and it has continued through the more sophisticated stages in its subsequent history. Although algebra and the use of coordinates (particularly "canonical coordinates") play an important role, a purely algebraic approach rapidly gets bogged down in massive and unenlightening formulae which have neither theoretical nor practical utility.

The right general context which emerged for the study of mechanics was that of symplectic geometry. This can be formally compared with the more familiar Riemannian geometry, where the metric tensor g_{ij} is symmetric: in symplectic geometry the analogous tensor is skew-symmetric. However, this analogy is superficial; the motivation, techniques, and applications are totally different. The basic example of symplectic geometry is R^{2n}, with coordinates $p_1, \dots, p_n, q_1, \dots, q_n$, where the basic datum is the exterior differential 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

This example arises in mechanics as the "phase-space" of a particle in R^n: the q_i are the position coordinates, and the p_i are the corresponding momenta. The key properties of ω are that it is *nondegenerate*, so that ω^n gives a volume form, and that it is *closed*, i.e., $d\omega = 0$. Abstracted out, this leads to the definition of a *symplectic manifold.*

A dynamical system on a symplectic manifold is defined by its Energy or Hamiltonian function H which naturally generates the Hamiltonian flow, describing how the system evolves in time (the vector field of the flow is dual to the 1-form dH, the duality being relative to ω).

The history of mechanics is also closely associated with the history of optics, particularly in the work of Hamilton. The way in which geometrical optics approximates the wave propagation of light is now understood in the framework of symplectic geometry. More generally the modern theory of linear partial differential operators makes extensive use of "microlocal" analysis, i.e., working with phase-space and its symplectic automorphisms. This is also the framework in which classical mechanics is seen as an approximation to a corresponding quantum-mechanical system. In fact, it was the advent of quantum mechanics and the realization that Hamilton's theory provided the right framework that led in this century to a general revival of interest in classical mechanics and symplectic geometry.

Traditionally conservation laws (e.g., of energy and momentum) have played an important role in mechanics, but their status was not fully understood until Emmy Noether showed how, in a Hamiltonian framework, continuous symmetries (invariance under Lie groups) lead quite generally to conservation laws. Thus translational invariance leads to the conservation of linear momentum, while rotational invariance leads to the conservation of angular momentum.

Conservation laws are exploited by fixing the values of the conserved quantities and then working with a reduced system. For example, fixing the linear momentum amounts to working relative to the centre of mass. This process can be treated quite generally in the context of a Lie group G acting on a symplectic manifold X. Under appropriate technical conditions one constructs a "symplectic quotient" $X//G$. Note that

$$\dim X//G = \dim X - 2\dim G.$$

It is important to realize that $X//G$ is only a subspace of the ordinary quotient X/G: it is the subspace obtained by fixing all the conserved quantities.

An important class of symplectic manifolds arises naturally in complex algebraic geometry and provides a motivation quite different from that of classical mechanics. In Hodge's fundamental work on harmonic forms and the homology of algebraic varieties, he appreciated the importance of the class of Kähler metrics. These are Hermitian metrics which preserve the complex structure (under parallel transport), and they are automatically symplectic. In fact, one could say that

$$\begin{aligned}\text{Algebraic Geometry} &\rightarrow \text{Kähler Geometry}\\ &= \text{Riemannian Geometry} \cap \text{Symplectic Geometry}.\end{aligned}$$

This link between algebraic geometry and symplectic geometry has proved to be very fruitful. For example, the symplectic quotient $X//G$ has been shown essentially to coincide with Mumford's complex algebraic quotient X/G^c, where G^c is the complexification of the compact Lie group G. For example, when $X = C^n$ and $G = U(n)$, acting by scalar multiplication, the quotient is the complex projective $(n-1)$-space. Moreover, this story extends naturally to the associated "quantizations". In fact, in the algebraic theory quantization came first as the theory of invariants; Mumford's geometric invariant theory came later.

One interesting by-product of this identification of symplectic quotients with algebrogeometric quotients has been a general procedure, due to Frances Kirwan, for computing the cohomology of such quotients.

Many interesting examples of symplectic manifolds are infinite-dimensional; and although these naturally have to be treated with appropriate analytic care, the symplectic formalism remains surprisingly useful. For example, Arnold pointed out that the flow of an ideal fluid could naturally be interpreted in terms of the infinite-dimensional manifold of volume-preserving diffeomorphisms of the domain. More recent examples have arisen in gauge theories. For instance, the space of G-connections on a closed Riemann surface is naturally an infinite-dimensional symplectic manifold, and its symplectic quotient (by the group of bundle automorphorphisms or "gauge transformations") turns out to be the moduli space algebraic geometers associated to the Riemann surface for classifying holomorphic G^c bundles.

A subplot to all the above has been that of hyperkähler geometry, related to quaternions in the way kähler geometry relates to the complex numbers. A hyperkähler manifold has three independent kähler (and so symplectic) structures and is a very special and rigid object. However, there are many interesting examples (including in particular 4-dimensional Einstein manifolds), and the symplectic quotient construction has a natural extension to the formation of hyperkähler quotients $X///G$ with a dimension formula

$$\dim X///G = \dim X - 4 \dim G.$$

Many examples of interest arise when X and G are both function spaces (and so infinite-dimensional) but $X///G$ is finite dimensional. For example, the moduli spaces of instantons on R^4 are all hyperkähler manifolds with this origin. Other examples, due to Kronheimer, are related to the theory of rational double points (Kleinian singularities) on algebraic surfaces.

Symplectic geometry is now under intensive study, partly motivated by ideas from theoretical physics. Gromov has developed a theory of pseudoholomorphic curves on symplectic manifolds, and Floer introduced the homology groups, now named after him, in the course of solving Arnold's conjecture about fixed points of symplectic diffeomorphisms. These fixed points are related to classical questions of closed orbits of Hamiltonian flows and have a genealogy going back to Poincaré and Birkhoff.

Marsden's book, based on a series of lectures for the London Mathematical Society, centres around symmetry and symplectic quotients. Many examples are given illustrating the utility and relevance of symplectic quotients. Mostly these examples are of a traditional nature involving one or more rigid bodies, and the relevant symmetries are rotations.

One topic treated at length is that of "phases" of the type discovered and popularized by Berry and Hannay. The simplest case consists of a bead sliding along a wire in the shape of an arbitrary closed planar loop. If the loop is rotated slowly through $360°$, one finds that the bead will, in its motion, be displaced by a certain time-interval (or phase) from what it would have been for a stationary loop. There are many other interesting examples, and they can be understood in terms of the holonomy of an associated fibre bundle (arising in the symplectic quotient construction).

Infinite-dimensional examples of Hamiltonian systems in which this Berry phase occurs include the soliton interactions in integrable equations of the KdV type—an extensive subject in its own right.

Another mechanical system where rotational symmetry is particularly helpful (and which Marsden treats in detail) is that of the "falling cat". How is it that a cat, dropped upside down from an appropriate height, will twist so as to land on its feet? It turns out that this is a subtle problem with an elegant solution, and it is part of the more general problem of the dynamics of deformable bodies and feed-back control.

Marsden's book is not a textbook but a series of lectures on various aspects of symmetry in dynamics. There is enough background to make the book reasonably self-contained, and there is a very thorough bibliography and an indication of where to go to pursue any of the more specialized topics. The style is readable and stimulating.

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Tube domains and the Cauchy problems, by Simon Gindikin. Translations of Mathematical Monographs, vol. 111, American Mathematical Society, Providence, RI, v + 132 pp., $78.00. ISBN 0-8218-4566-7

This book, which is part of the author's thesis, deals with research from the early sixties carried out by a circle of former students of Gelfand. There are two parts. The first one deals with a specific class of partial differential operators, the second with certain generalized gamma functions associated with homogeneous cones.

The classical classes of differential operators—elliptic, parabolic, and hyperbolic—are all of second order, and their importance stems from physics. The study of higher-order constant coefficient operators for their own sake was made possible by the theory of distributions. In the middle fifties, three classes of such operators P had been characterized by intrinsic properties as follows: elliptic (all solutions of $Pu = 0$ analytic), hypoelliptic (all such solutions infinitely differentiable), and hyperbolic (fundamental solution with support in a cone). In all cases there are corresponding properties of the characteristic polynomials.

The first part of Gindikin's monograph is a second generation effort in the same direction. The starting point is a separation of variables in time t and space $x \in R^n$ and the corresponding Cauchy problem. Let $D_t = \partial/i\partial t$ and $D_x = \partial/i\partial x$ be the imaginary gradients so that $P(\tau, \xi)$ is the characteristic polynomial of $P(D_t, D_x)$. The author considers a class of operators for which $P(\tau, \xi)$ does not vanish in some tubular region

$$T\colon \operatorname{Im}\tau < -\chi(\operatorname{Im} x) - \text{const}$$

where χ is a fixed, finite convex function. Operators in this class have inverses given by the Fourier-Laplace transform and operate on classes of functions whose size in the x-directions is controlled by the dual of χ. The class itself is invariant under complex translations. With the added condition that the