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Multidimensional residues and their applications by A. K. Tsikk. American Mathematical Society, Providence, RI, 1992, x+187 pp., \$119.00. ISBN 0-8218-4560-8

In the classical theory of functions of one complex variable the residue theorem plays a central role. Together with its variants such as the Poisson-Jensen formula, the residue theorem may, in fact, be taken as the primary tool for developing the subject in almost all of its aspects. This applies not only to classical analytic function theory but also to algebraic function theory—i.e., the study of compact Riemann surfaces. For example, the Riemann-Roch Theorem may be expressed as saying that the constraints imposed by the residue theorem are necessary and sufficient for a set of principal parts to be realized by a global meromorphic function.

The present monograph may be understood as seeking to develop the residue concept as the central object in a number of aspects of several complex variables. I have used the word *concept* because in several variables there are, in fact, a number of related definitions of the residue, each with its own theorem and uses. Among these, however, one stands out as the most fundamental: Let $f_1(z), \dots, f_n(z)$, $g(z)$ be complex analytic functions of $z = (z^1, \dots, z^n)$ defined in a neighborhood U of the origin, and assume that the analytic equations

$$f_1(z) = \cdots = f_n(z) = 0$$

have the origin as an isolated common zero. For the meromorphic n-form

$$\omega = \frac{g(z)dz^1 \wedge \cdots \wedge dz^n}{f_1(z) \cdots f_n(z)}$$

one defines the *residue* to be given by the integral

$$\operatorname{Res}_0 \omega = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \omega \tag{1}$$

where Γ is the n-cycle given by

$$\Gamma = \{z : |f_i(z)| = \epsilon_i \text{ for } i = 1, \dots, n\}$$

with orientation $df_1 \wedge \cdots \wedge df_n \geq 0$. (We shall also sometimes write $\operatorname{Res}_0(g)$ for (1).) Although used by analysts for many years, the basic properties of the residue were first formulated and organized by Grothendieck, who defined $\operatorname{Res}_0 \omega$ purely algebraically and called it the residue symbol $\operatorname{Res}\left[\begin{smallmatrix} g\,dz \\ f_1 \cdots f_n \end{smallmatrix}\right]$.[1] Among these properties perhaps the most basic is the *local duality theorem*: *Let*

[1]The fact that "Res" may be defined purely algebraically is suggested by the result that the distribution $g \to \operatorname{Res}_0(g)$ is a linear combination of derivatives of δ-functions at the origin. Typically, Grothendieck showed that the residue symbol is the unique function of the data $\{g; f_1, \dots, f_n; z^1, \dots, z^n\}$ having certain functorial properties.

$\mathscr{O}$ be the local ring of analytic functions defined in some neighborhood of the origin and $\mathscr{I} \subset \mathscr{O}$ the ideal generated by $f_1, \ldots, f_n$. Then $\operatorname{Res}[\frac{g\,dz}{f_1 \cdots f_n}]$ *depends only on* $g \bmod \mathscr{I}$ *and the pairing*

$$\mathscr{O}/\mathscr{I} \otimes_{\mathbb{C}} \mathscr{O}/\mathscr{I} \longrightarrow \mathbb{C}$$

given by

$$g \otimes h \dashrightarrow \operatorname{Res}\left[\frac{gh\,dz}{f_1 \cdots f_n}\right]$$

is nondegenerate.

Following some preliminaries in Chapter I, the residue (1) is introduced and its basic properties are derived in Chapter II. A number of applications are given, among which it is shown that the geometric and algebraic properties of the finite holomorphic mapping $f: U \longrightarrow \mathbb{C}^n$ given by

$$w = f(z) = (f_z(z), \ldots, f_n(z)) \tag{2}$$

may be understood by the residue. Basically, if we consider (2) as an equation with solutions $z_1(w), \ldots, z_\mu(w)$ for each sufficiently small w, then the degree μ is constant; and for any $g(z)$ the elementary symmetric functions of $g(z_1(w)), \ldots, g(z_\mu(w))$ are expressed by suitable residues, thus in principle leading to an essentially complete description of the ring extension $[\mathscr{O}_w : \mathscr{O}_z]$.

Among the applications of the local duality theorem are many of the classical results on regular ideals dating from Macaulay. For example, the statement that if $g(0) = 0$, then $g^\mu \in \mathscr{I}$ gives the sharp form of the Hilbert nullstellensatz for the ideal generated by $f_1, \ldots, f_n$. Later in Chapter IV further applications of the residue concept are shown to include most of the traditional basic results in local analytic geometry, such as the Weinstrass preparation and division theorems, various extension results, and the standard integral representation theorems (Boohner-Martenelli, Weil, etc.).

There are also a number of global residue theorems, both for $\mathbb{C}^n$ and for compact complex manifolds. For example, in $\mathbb{C}^n$ the global residue theorem may be used to derive Bezout's theorem[2] and the *Euler-Jacobi formula*: If $f_1(z), \ldots, f_n(z)$ are polynomials having only simple common roots not on the hyperplane at infinity, then for any polynomial $g(z)$ of degree less than $\deg f_1 + \cdots + \deg f_n - n$

$$\sum_{f(a)=0} \operatorname{Res}\left(\frac{g}{J_f}\right) = 0$$

where the sum is over the common roots the $f_i(z)$ and J_f is the Jacobian $df_1 \wedge \cdots \wedge df_n/dz^1 \wedge \cdots \wedge dz^n$. When $n = 2$ and f_1, f_2 are cubics, this result implies that any cubic passing through eight of the nine points of intersection must pass through the ninth as well, one of the earliest results in algebraic geometry.[3] Modern applications of the $\sum$ Residues $= 0$ concept may be given to the study of vector bundles, especially on algebraic surfaces.

[2]This is analogous to using the residue theorem to derive the fundamental theorem of algebra in elementary 1-variable theory.

[3]Since cubics depend on nine parameters, this shows that there are constraints beyond dimension counts on a set of points in $\mathbb{C}^2$ to be the common zeros of two polynomials. Referring to footnote 1, the infinitesimal statement is that for $n \geq 2$ not every δ-function supported at the origin is given by a residue.

Chapter III of this book is devoted to a study of *residue currents* associated to complete intersections of positive dimension. The theory here, due to Coleff, Herria, and others, has numerous applications to local analytic geometry generalizing many of the properties of zero-dimensional complete intersections. The analysis is naturally somewhat more involved and requires the careful definition and use of principal values. In the case of codimension-one hypersurfaces contact is made with the "tube-over-cycle" residue map used by J. Leray in his study of fundamental solutions of hyperbolic partial differentiality equations.

In conclusion, what may we conclude about residues in several complex variables? Although the subject did not develop in this way, the monograph provides a convincing demonstration that residues may be used in an effective manner to develop much of local analytic geometry. Although they are less successful in dealing with ideals that are not local complete intersections, this is more than offset by the efficiency and eloquence with which residues apply to the wide variety of topics included in this book. One could well imagine an appealing introductory textbook in several complex variables that takes the residue (1) as its point of departure and for which the curent book would provide a valuable source of interesting applications.

A number of topics in global analytic and algebraic geometry are not covered in the monograph under review, so the reader may not get a sense of the extent to which residues may eventually have a significance in higher dimension comparable to their profound use in compact Riemann surface thoery. Among topics we would mention are the relationship between residues and Hodge theory and the various extensions of Abel's theorem. Regarding the latter, there is a recent paper by Henkin (G. M. Henkin, *The Abel-Radon Transform And Several Complex Variables*) in which residues are used via the "Abel transform" to give a far-reaching extension of the classical Abel's theorem, which in its original formulation was based on the elementary observation that $\mathrm{Res}_0\, \omega(\xi)$ is meromorphic in a parameter ξ if $\omega(\xi)$ itself is.

In summary, the present volume is useful and well presented, although a substantially expanded index would have been helpful; however, the reviewer would have preferred more emphasis on global results illustrating the priniciple that "$\sum \mathrm{Res} = 0$ is an analytic manifestation of compactness", a principle that has yet to be fully explored in applications.

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Introduction to multidimensional integrable equations, by B. G. Konopelchenko. Plenum Publishing, New York, 1992, x + 292 pp, $85.00. ISBN 0-306-44220-5

The field of soliton theory and the related methods of solution of the underlying nonlinear evolution equations, referred to as the Inverse Scattering