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A SUBSEQUENCE PRINCIPLE CHARACTERIZING BANACH SPACES CONTAINING c_0

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ABSTRACT. The notion of a strongly summing sequence is introduced. Such a sequence is weak-Cauchy, a basis for its closed linear span, and has the crucial property that the dual of this span is not weakly sequentially complete. The main result is:

Theorem. *Every non-trivial weak-Cauchy sequence in a (real or complex) Banach space has either a strongly summing sequence or a convex block basis equivalent to the summing basis.*

(A weak-Cauchy sequence is called *non-trivial* if it is *non-weakly convergent.*) The following characterization of spaces containing c_0 is thus obtained, in the spirit of the author's 1974 subsequence principle characterizing Banach spaces containing ℓ^1.

Corollary 1. *A Banach space B contains no isomorph of c_0 if and only if every non-trivial weak-Cauchy sequence in B has a strongly summing subsequence.*

Combining the c_0-and ℓ^1-theorems, one obtains

Corollary 2. *If B is a non-reflexive Banach space such that X^* is weakly sequentially complete for all linear subspaces X of B, then c_0 embeds in B.*

1. INTRODUCTION

When does a general Banach space contain one of the classical sequence spaces? In particular, when does it contain one of the non-reflexive ones, ℓ^1 (the space of absolutely summable sequences) or c_0 (the space of sequences vanishing at infinity)?

In 1974, the following subsequence dichotomy was established by the author for real scalars [R1] and refined by L. E. Dor to cover the case of complex scalars [Do] (cf. also [R2] for a general exposition).

Theorem 1.0. *Every bounded sequence in a real or complex Banach space has either a weak-Cauchy subsequence or a subsequence equivalent to the standard ℓ^1-basis.*

The following characterization is an immediate consequence.

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Corollary 1 [R1]. *A real or complex Banach space B contains no isomorph of ℓ^1 if and only if every bounded sequence in B has a weak-Cauchy subsequence.*

Using standard functional analysis, one easily obtains

Corollary 2. *If B is a non-reflexive Banach space so that B is weakly sequentially complete, then ℓ^1 embeds in B.*

To obtain analogous results to characterize spaces containing c_0, the following concept is introduced:

Definition. A sequence (b_j) in a Banach space is called *strongly summing*, or (s.s.), if (b_j) is a weak-Cauchy basic sequence so that whenever scalars (c_j) satisfy $\sup_n \|\sum_{j=1}^n c_j b_j\| < \infty$, the series $\sum c_j$ converges.

A simple permanence property: let (b_j) be an (s.s.) basis for a Banach space B, and let (b_j^*) be its biorthogonal functionals in B^*, i.e., $b_j^*(b_i) = \delta_{ij}$ for all i and j. Then $(\sum_{j=1}^n b_j^*)_{n=1}^\infty$ is a non-trivial weak-Cauchy sequence in B^*; hence B^* *fails* to be weakly sequentially complete. (A weak-Cauchy sequence is called *non-trivial* if it is *non-weakly convergent.*)

The following is the main result of this announcement.

Theorem 1.1. *Every non-trivial weak-Cauchy sequence in a (real or complex) Banach space has either a strongly summing subsequence or a convex block basis equivalent to the summing basis.*

Remark. The two alternatives of Theorem 1.1 are easily seen to be mutually exclusive.

Recall that the *summing basis* denotes the unit vectors basis for the space of all converging series of scalars, denoted Se, endowed with the norm $\|(c_j)\|_{\mathrm{Se}} = \sup_n |\sum_{i=1}^n c_i|$. It is a standard, simple result that Se is isomorphic to c_0, and hence we obtain the following analogue of Corollary 1.

Corollary 1.1. *A Banach space B contains no isomorph of c_0 if and only if every non-trivial weak-Cauchy sequence in B has an* (s.s.) *subsequence.*

Combining the c_0- and ℓ^1-theorems, we obtain the following result, analogous to Corollary 2.

Corollary 1.2. *If B is a non-reflexive Banach space such that X^* is weakly sequentially complete for all linear subspaces X of B, then c_0 embeds in B.*

We note that the hereditary hypothesis in Corollary 1.2 is crucial. Indeed, J. Bourgain and F. Delbaen construct in [Bo-De] a Banach space B containing *no* isomorph of c_0, such that B^* is isomorphic to ℓ^1 (so of course B^* is weakly sequentially complete).

To prove the main result, Theorem 1.1, we develop various permanence properties of strongly summing sequences and then employ some transfinite invariants for general discontinuous functions introduced by A. S. Kechris and A. Louveau [AL]. These are used to characterize (complex) differences of bounded semi-continuous functions, which are essentially involved in the proof. The core of the argument is then a real-variables theorem concerning a subsequence refinement principle for a uniformly bounded sequence of continuous functions, converging pointwise to a function which is *not* such a difference.

In Section 2, we outline the proof of the main result, reducing to a qualitative version of the real-variables principle. In Section 3, we formulate the Kechris-Louveau invariants, which are termed here the "positive transfinite oscillations", as well as some related new invariants, called the "transfinite-oscillations". The latter yield a surprising norm-identity on the Banach algebra of (complex) differences of bounded semi-continuous functions (Theorem 3.1). Finally, we formulate a quantitative version of the real-variables principle (Theorem 3.3), which yields the qualitative version and thus the proof of the main result. All the results stated here, as well as further complements, are given in full detail in [R3].

2.

Throughout, we use standard Banach space terminology; here is a quick review of some of the necessary concepts. Let (b_j) be a sequence in a Banach space B. (b_j) is called a *basic sequence* if (b_j) is a basis for its closed linear span, denoted $[b_j]$. Equivalently, the b_j's are non-zero and there is a positive λ so that $\|\sum_{i=1}^k c_i b_i\| \leq \lambda \|\sum_{i=1}^n c_i b_i\|$ for all $1 \leq k < n$ and scalars $c_1, \dots, c_n$; if λ works, we call (b_j) λ-basic. A sequence (u_j) in B is called a *block basis* of (b_j) if there exist integers $0 \leq n_1 < n_2 < \cdots$ and scalars $c_1, c_2, \dots$ so that $u_j = \sum_{i=n_j+1}^{n_{j+1}} c_i b_i$ for all $j = 1, 2, \dots$; (u_j) is called a *convex block basis* if the c_i's can be chosen to satisfy $c_i \geq 0$ for all i and $\sum_{i=n_j+1}^{n_{j+1}} c_i = 1$ for all j. Sequences (x_i) and (y_i) in (possibly different) Banach spaces are called *equivalent* if there exists an *isomorphism* (i.e., a bounded linear invertible) operator $T : [x_i] \to [y_i]$ with $Tx_i = y_i$ for all i. If (b_j) is a given basic sequence, (b_j^*) denotes the sequence of its biorthogonal functionals in $[b_j]^*$, the dual of the closed linear span of the b_j's.

We first indicate the reduction of the proof of Theorem 1.1 to a "classical real variables" setting. For K a compact metric space, $D(K)$ denotes the set of all (complex) differences of bounded semi-continuous functions on K; that is, $f \in D(K)$ if and only if there are bounded lower semi-continuous functions $u_1, \dots, u_4$ on K so that $f = (u_1 - u_2) + i(u_3 - u_4)$: equivalently (by results of Baire), if and only if there exists a sequence (φ_j) in $C(K)$ (the space of continuous functions on K) with $\sup_{k \in K} \sum |\varphi_j(k)| < \infty$ and $f = \sum \varphi_j$ pointwise.

The next result follows from refinements of arguments in [Bes-P]; see Corollary 3.1 of [HOR] (also cf. Proposition 1.7 of [R3]).

Proposition 2.1. *Let K be a compact metric space, $f : K \to \mathbb{C}$ discontinuous, and (f_n) uniformly bounded in $C(K)$ with $f_n \to f$ pointwise. Then f is in $D(K)$ if and only if (f_n) has a convex block basis equivalent to the summing basis.*

The c_0-theorem then follows immediately from Proposition 2.1 and the following result (in both Proposition 2.1 and Theorem 2.2, the ambient Banach space B is $C(K)$).

Theorem 2.2. *Let K be a compact metric space and (f_n) be a uniformly bounded sequence in $C(K)$ which converges pointwise to a function f. If f is not in $D(K)$, then (f_n) has an* (s.s.) *subsequence.*

To prove Theorem 2.2, we need some natural permanence properties of (s.s.) sequences, including the following companion notion.

Definition. A basic sequence (e_j) in a Banach space is called *coefficient converging*, or (c.c.), if $(\sum_{j=1}^n e_j)$ is a weak-Cauchy sequence so that whenever scalars (c_j) satisfy $\sup_n \|\sum_{j=1}^n c_j e_j\| < \infty$, the sequence (c_j) converges.

In the sequel, given sequences (b_j) and (e_j) in a Banach space, (e_j) is called the *difference sequence* of (b_j) if $e_1 = b_1$ and $e_j = b_j - b_{j-1}$ for all $j > 1$.

Proposition 2.3. *Let (b_j) be a given sequence in a Banach space.*

(a) *(b_j) is* (s.s.) *if and only if its difference sequence is* (c.c.).
(b) *If (b_j) is* (s.s.), *every convex block basis of (b_j) is also* (s.s.).
(c) *If (b_j) is a basic sequence, then (b_j) is* (s.s.) *if and only if (b_j^*) is* (c.c.).

We note that (c) yields that if (b_j) is (s.s.), $[b_j]^*$ is not weakly sequentially complete, for it follows easily that if (e_j) is (c.c.), $(\sum_{j=1}^n e_j)$ is a non-trivial weak Cauchy sequence. (b) or (c) combined with 2.1 may be used to show the alternatives in the c_0-theorem are mutually exclusive.

To obtain (s.s.)-sequences, we first note that every non-trivial weak-Cauchy sequence in a Banach space has an (s)-subsequence; i.e., a weak-Cauchy basic sequence which dominates the summing basis (cf. [HOR]). Then we use the following quantitative result, which follows by diagonalization and an argument of S. Bellenot [Be].

Lemma 2.4. *Let (f_j) be an* (s)*-sequence in a Banach space. Then (f_j) has an* (s.s.)*-subsequence provided for every $\varepsilon > 0$ and subsequence (g_j) of (f_j), there is a subsequence (b_j) of (g_j) with difference sequence (e_j) so that whenever (c_j) is a sequence of scalars with $c_j = 0$ for infinitely many j and $\|\sum_{j=1}^n c_j e_j\| \le 1$ for all n, $\overline{\lim}_{j\to\infty} |c_j| \le \varepsilon$.*

We now arrive at the real-variables core of the proof.

Theorem 2.5. *Let K be a compact metric space and (f_n) be a uniformly bounded sequence in $C(K)$ which converges pointwise to a function f which is not in $D(K)$. There exists a c with $|c| = 1$ so that given $M < \infty$, $\kappa > 0$, there is a subsequence (b_j) of (cf_j) with difference sequence (e_j) so that given $0 < m_1 < m_2 < \cdots$ an infinite sequence of integers, there exists a t in K and an integer k with*

(1) $\sum_{j=1}^k \operatorname{Re} e_{m_{2j}}(t) > M$;
(2) $\operatorname{Re} e_{m_{2j}}(t) > 0$ *for all* $1 \le j \le k$;
(3) $\sum_{1<i\notin\{m_1, m_2, \dots, m_k\}} |e_i(t)| < \kappa$.

Sketch of proof of Theorem 2.2, *real scalars.* We easily reduce to the case where (f_n) is an (s)-sequence satisfying the conclusion of Theorem 2.5 with $c = 1$. It then follows that there are finite λ, τ so that if (b_j) is a subsequence of (f_j) with difference sequence (e_j), then

$$(1) \qquad (e_j) \text{ is } \lambda\text{-basic and } \|e_j^*\| \le \tau \quad \text{for all } j.$$

Now let $\varepsilon > 0$, (f_j') be a subsequence of (f_j), $\kappa = 1$, $M = \frac{1+\tau+\lambda}{\varepsilon}$, and (b_j) be a subsequence of (f_j') satisfying the conclusion of Theorem 2.5. It follows

that the difference sequence (e_j) of (b_j) satisfies the conclusion of Lemma 2.4, thus completing the proof. □

3.

To obtain Theorem 2.5, we require certain transfinite invariants for a general discontinuous function. We define the *transfinite oscillations* of a given function $f : K \to \mathbb{C}$; these are similar to invariants previously defined in [KL], which we term here the *positive transfinite oscillations.* Fix K a compact metric space. For $f : K \to [-\infty, \infty]$ an extended real-valued function, Uf denotes the upper semi-continuous envelope of f; $Uf(x) = \overline{\lim}_{y\to x} f(y)$ for all $x \in K$. (We use non-exclusive lim sups; thus $\overline{\lim}_{y\to x} f(y) = \inf_U \sup f(U)$, the infimum over all open neighborhoods U of x.)

Definition. Let $f : K \to \mathbb{C}$ be a given function, K a compact metric space, and α a countable ordinal. We define the αth oscillation of f, $\operatorname{osc}_\alpha f$, by induction, as follows: set $\operatorname{osc}_0 f \equiv 0$. Suppose $\beta > 0$ is a countable ordinal and $\operatorname{osc}_\alpha f$ has been defined for all $\alpha < \beta$. If β is a successor, say $\beta = \alpha + 1$, we define

$$\widetilde{\operatorname{osc}}_\beta f(x) = \overline{\lim}_{y\to x}\left(|f(y) - f(x)| + \operatorname{osc}_\alpha f(y)\right) \quad \text{for all } x \in K .$$

If β is a limit ordinal, we set

$$\widetilde{\operatorname{osc}}_\beta f = \sup_{\alpha<\beta} \operatorname{osc}_\alpha f .$$

Finally, we set $\operatorname{osc}_\beta f = U\widetilde{\operatorname{osc}}_\beta f$.

The *positive* α^{th} *oscillation* of a real-valued function f, $v_\alpha f$, is defined in exactly the same way, except that the absolute value signs are deleted. These are given in [KL], with a different terminology and equivalent formulation, where it is established that *f is in $D(K)$ if and only if $v_\alpha f$ is a bounded function for all $\alpha < w_1$. Moreover, when this happens, there is an $\alpha < w_1$ with $v_\alpha f = v_{\alpha+1} f$. Then writing $f = u_\alpha - v_\alpha$, u_α is upper semi-continuous.*

We obtain a similar result, using the transfinite oscillations, which yields a rather surprising norm identity. We define a norm $\|\cdot\|_D$ on $D(K)$ by

$$\|f\|_D = \inf\left\{\sup_{k\in K}\sum_j |\varphi_j(k)| : (\varphi_j) \text{ in } C(K) \text{ with } \sum \varphi_j = f\right\} .$$

Theorem 3.1. *Let $f : K \to \mathbb{C}$ be a bounded function. Then $f \in D(K)$ if and only if $\operatorname{osc}_\alpha f$ is a bounded function for all $\alpha < w_1$. When this occurs, there is a countable α with $\operatorname{osc}_\alpha f = \operatorname{osc}_{\alpha+1} f$. If f is real valued, then setting $\lambda = \| |f| + \operatorname{osc}_\alpha f\|_\infty$, we have that $\|f\|_D = \lambda$, and if $u = (\lambda - \operatorname{osc}_\alpha f + f)/2$, $v = (\lambda - \operatorname{osc}_\alpha f - f)/2$, then u, v are non-negative lower semi-continuous functions with $f = u - v$ and $\|f\|_D = \|u - v\|_\infty$.*

It then follows that the "inf" in the definition of $\|f\|_D$ is actually attained (for real-valued f).

Although only the positive transfinite oscillations are actually needed for proving the c_0-theorem, we prefer to use the transfinite oscillations, since they appear more natural in studying the Banach space structure of $D(K)$ and related objects (see [R4]). The connection between the oscillations is given by the following simple result.

Proposition 3.2. *Let $f : K \to \mathbb{R}$ be a given function and α a countable ordinal. Then*

$$v_\alpha(f) \le \operatorname{osc}_\alpha f \le v_\alpha(f) + v_\alpha(-f) .$$

Theorem 2.5 then follows from Theorem 3.1 and Proposition 3.2 (or the cited results in [KM], and the following quantitative real-variables result.

Theorem 3.3. *Let (f_j) be a uniformly bounded sequence of complex-valued continuous functions on K a compact metric space, converging pointwise to a function f. Let α be a countable ordinal and $x \in K$ be given with $0 < v_\alpha(\varphi)(x) \overset{\mathrm{df}}{=} \lambda < \infty$ where $\varphi = \operatorname{Re} f$. Let $\mathscr{U}$ be an open neighborhood of x and $0 < \eta < 1$ be given. There exists (b_j) a subsequence of (f_j) with the following properties: Given $1 = m_1 < m_2 < \cdots$ an infinite sequence of integers, there exist k, points $x_1, \dots, x_{2k-1}, x_{2k} \overset{\mathrm{df}}{=} t$ in $\mathscr{U}$, and positive numbers $\delta_1, \dots, \delta_k$ so that:*

(1) $\varphi(x_{2j}) - \varphi(x_{2j-1}) > (1-\eta)\delta_j$ *for all* $1 \le j \le k$,
(2) $(1+\eta)\lambda > \sum_{j=1}^{k} \delta_j > (1-\eta)\lambda$,
(3) $\sum_{m_j \le i < m_{j+1}} |b_i(t) - f(x_j)| < \eta\delta_{[\frac{j+1}{2}]}$ *for all* $1 \le j \le 2k-1$,
(4) $\sum_{i \ge m_{2k}} |b_i(t) - f(t)| < \eta\,\delta_k$.

The proof of Theorem 3.3 is given by transfinite induction, and the formulation in terms of arbitrary open neighborhoods $\mathscr{U}$ is crucial, although this is not used in the proof of Theorem 2.5. The argument is easily reduced to the "α to $\alpha+1$" case.

Sketch of proof of the induction step. We suppose the result proved for α and assume $0 < v_\alpha(\varphi)(x) < v_{\alpha+1}(\varphi)(x) \overset{\mathrm{df}}{=} \beta$. Now given $\eta > 0$ and $\mathscr{U}$ an open neighborhood of x, we obtain the existence of positive numbers $\bar{\lambda}$ and δ with $(1-\eta)\beta < \bar{\lambda} + \delta < (1+\eta)\beta$, $x_1 \in \mathscr{U}$, and a subsequence (b_j) of (f_j), so that given $1 < r \le s$, there is an open set $\mathscr{V} \subset \mathscr{U}$ and an $x_2 \in \mathscr{V}$ with

$$\varphi(x_2) - \varphi(x_1) > (1-\eta)\delta ; \tag{2}$$

$$\sum_{1 \le i < r} |b_i(t) - f(x_1)| < \eta\delta \quad \text{for all } t \in \mathscr{V} ; \tag{3}$$

$$\sum_{r \le i \le s} |b_i(t) - f(x_2)| < \eta\delta \quad \text{for all } t \in \mathscr{V} ; \tag{4}$$

$$\underline{\lambda} \le \lambda < (1+\eta)\beta - \delta , \quad \text{where } \lambda = v_\alpha(\varphi)(x_2) . \tag{5}$$

The numbers $\bar{\lambda}$ and δ are obtained using the definition of $v_{\alpha+1}(\varphi)$, and (b_i) is obtained from the proof of the "$\alpha = 1$" case. Finally, after a further refinement, using the inductive hypothesis, we obtain

$$\begin{cases} (b_i)_{i>s} \text{ satisfies the conclusion of Theorem 3.3} \\ \text{for the } \alpha\text{-case, with "}\mathscr{U}\text{"} = \mathscr{V}, \text{ "}x\text{"} = x_2. \end{cases} \tag{6}$$

Now thanks to (2), (5) and the fact that the inequalities in (3), (4) hold for *all* $t \in \mathscr{V}$, we obtain that (b_i) satisfies the conclusion of Theorem 3.3 for the $(\alpha+1)$-case. □

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