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## THE FLAG-TRANSITIVE TILDE AND PETERSEN-TYPE GEOMETRIES ARE ALL KNOWN

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**ABSTRACT.** We announce the classification of two related classes of flag-transitive geometries. There is an infinite family of such geometries, related to the nonsplit extensions  $3^{l_2^2} \cdot \text{Sp}_{2n}(2)$ , and twelve sporadic examples coming from the simple groups  $M_{22}$ ,  $M_{23}$ ,  $M_{24}$ ,  $He$ ,  $Co_1$ ,  $Co_2$ ,  $J_4$ ,  $BM$ ,  $M$  and the nonsplit extensions  $3 \cdot M_{22}$ ,  $3^{23} \cdot Co_2$ , and  $3^{4371} \cdot BM$ .

### 1. INTRODUCTION

An *incidence system*  $\mathcal{G}$  is a set of elements, each having a particular type (or coloring), on which a reflexive, symmetric, binary incidence relation is defined. Two different elements of the same type can never be incident. A *flag* is a set of pairwise incident elements. The *residue* of a flag  $F$ , denoted  $\text{res}(F)$ , is the incidence system naturally formed by all the elements not in  $F$ , which are incident to all the elements from  $F$ . The rank of  $\mathcal{G}$  is the number of different types presented in  $\mathcal{G}$ . An incidence system  $\mathcal{G}$  is *connected* if the graph on  $\mathcal{G}$  defined by the incidence is connected.

A *geometry* is an incidence system for which two further properties hold: (1) every maximal flag contains elements of all types; (2) every residue of rank at least 2 (including the geometry itself) is connected. The latter condition is known as *strong* or *residual connectedness*. Both these properties are inductive, i.e., the residues of geometries are themselves geometries. This sort of inductiveness was used in the definition of the diagram of a geometry. An example of such a diagram can be found below. The vertices of a diagram are all the types of the corresponding geometry (geometries). If you remove from the diagram all the types presented in a flag  $F$  and all the edges on these types, then the remaining diagram will describe  $\text{res}(F)$ ; in particular, the edges of the diagram describe the rank 2 residues of the geometry. As the type set we usually take  $\{1, 2, \dots, n\}$ , where  $n$  is the rank and the types in the diagram increase from

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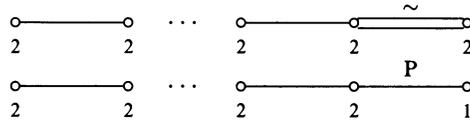
left to right. For more information about the axioms and the properties of the geometries and diagrams see [Bue1, Tit2].

A *flag-transitive* automorphism group  $G$  is a group of permutations on a geometry  $\mathcal{G}$ , which preserves types and the incidence, and, moreover, acts transitively on the set of maximal flags of  $\mathcal{G}$ . In this situation we also call  $\mathcal{G}$  *flag-transitive*. We often say that  $\mathcal{G}$  is the geometry of the group  $G$  and denote it  $\mathcal{G} = \mathcal{G}(G)$ .

The systematic study of the geometries of sporadic simple groups was initiated in [Bue1]. The idea was to develop a geometric theory of sporadic groups close to Tits's theory of buildings for the Chevalley groups [Tit1, Tit2]. In the latter case the diagrams of the buildings naturally correspond to the Dynkin diagrams of the corresponding Chevalley groups, and just like the Dynkin diagrams, the geometric diagrams of the buildings contain enough information to reconstruct the buildings and classify them.

By now, examples of flag-transitive geometries are known for all twenty-six sporadic groups. The diagrams of these geometries contain many new edges. Some (but not many) of the sporadic geometries were characterized by their diagrams, usually under the flag-transitivity assumption (cf. [Pas, Bue2]).

In this paper we announce the classification of the flag-transitive geometries belonging to the following two types of diagrams:



where  $\overset{\sim}{\circ} \text{---} \circ$  (resp.  $\circ \text{---} \overset{\text{P}}{\circ}$ ) denotes the triple cover of the generalized quadrangle for  $\text{Sp}_4(2)$  (resp. the geometry of edges and vertices of the Petersen graph). We call these geometries, respectively, *tilde* (for short,  $T$ -) and *Petersen-type* (for short,  $P$ -) geometries.

We present the list of  $T$ - and  $P$ -geometries and some of their properties in Table 1. In the fourth column we indicate the residue of an element of type 1; this defines a tree structure on the set of  $T$ - and  $P$ -geometries. Other entries will be explained later.

**Main Theorem.** *Every flag-transitive  $T$ - or  $P$ -geometry is isomorphic to one of the finite geometries listed in Table 1.*

For a flag-transitive group, by analogy with the case of buildings, the stabilizers of nonempty flags are called the *parabolic* subgroups. The rank of a parabolic is the rank of the residue of the corresponding flag. Let  $F$  be a maximal flag, and let  $M_1, M_2, \dots, M_n$  be the stabilizers of the elements from  $F$  ( $M_i$  is the stabilizer of the element of type  $i$ ). These stabilizers form the *amalgam of maximal parabolic subgroups*. By *amalgam* we understand just a collection of groups with common identity element and with the group operations consistent on intersections. The amalgam of maximal parabolics defines all the proper residues of the geometry, and, in particular, it defines the diagram. The flag-transitive geometries giving rise to the same amalgam are connected with each other by the operations of taking covers/quotients. A *covering* is a surjective

TABLE 1. *T*- and *P*-geometries

#	rank	Aut $\mathcal{G}$	Res.	2-cov.	Subgeom.	Nat. Reps.
$T_0$	2	$3 \cdot \text{Sp}_4(2)$	—	—	$P_0$	6+5
$T_0(n)$	$n > 2$	$3^{f_{2^2}^{n/2}} \cdot \text{Sp}_{2n}(2)$	$T_0(n-1)$	—	—	$2^n(2^n-1) + (2n+1)$
$T_1$	3	$M_{24}$	$T_0$	—	$P_1$	11
$T_2$	3	$He$	$T_0$	—	—	52
$T_3$	4	$Co_1$	$T_1$	—	$P_4$	24
$T_4$	5	$M$	$T_3$	—	$P_7$	—
$P_0$	2	$S_5$	—	—	—	6
$P_1$	3	Aut $M_{22}$	$P_0$	$P_2$	$\text{Sp}_4(2)$	11
$P_2$	3	$3 \cdot \text{Aut } M_{22}$	$P_0$	—	$T_0$	12+11
$P_3$	4	$M_{23}$	$P_1$	—	$A_7$	—
$P_4$	4	$Co_2$	$P_1$	$P_5$	$\text{Sp}_6(2)$	23
$P_5$	4	$3^{23} \cdot Co_2$	$P_2$	—	$T_0(3)$	0+23
$P_6$	4	$J_4$	$P_2$	—	$T_1$	—
$P_7$	5	$BM$	$P_4$	$P_8$	$\text{Sp}_8(2)$	—
$P_8$	5	$3^{4371} \cdot BM$	$P_5$	—	$T_0(4)$	—

morphism which is an isomorphism on every proper residue. Similarly, an *s-covering* is a surjective morphism which is an isomorphism on every residue up to rank *s*. There is a number of nontrivial 2-coverings between our geometries; they are shown in the fifth column of Table 1.

Within our approach the proof of the Main Theorem can be formally divided into two steps. First we classify possible amalgams of maximal parabolic subgroups (local characterization). The principal result is that every such amalgam comes from a known example. Second we determine the geometries/groups corresponding to the known amalgams. It amounts to proving that the final list of geometries is closed with respect to taking flag-transitive covers/quotients. It turned out that the original list of geometries was incomplete, and so we also faced a problem of constructing new *T*- and *P*-geometries related to strange nonsplit extensions of some sporadic and serial simple groups.

## 2. THE GEOMETRIES

The *T*-geometries  $\mathcal{G}(M_{24})$ ,  $\mathcal{G}(He)$ ,  $\mathcal{G}(Co_1)$ , and  $\mathcal{G}(M)$  were constructed in [RSt] as minimal parabolic geometries for the corresponding groups using their maximal parabolic geometries from [RSm1]. The *P*-geometries  $\mathcal{G}(M_{22})$ ,  $\mathcal{G}(3 \cdot M_{22})$ ,  $\mathcal{G}(M_{23})$ ,  $\mathcal{G}(Co_2)$ , and  $\mathcal{G}(BM)$  were constructed in [Ivn1] in terms of graphs. The truncations of these geometries by the elements of maximal type coincide with the minimal parabolic geometries from [RSt].

Suppose  $G < \text{Aut } H$ , where  $H$  is a group, and suppose  $H$  contains a subgroup  $E \cong 2^n$ , such that  $N_G(E)$  induces on  $E$  the whole group  $L_n(2)$ . Define an incidence system  $\mathcal{G} = \mathcal{G}(G, H, E)$  whose elements are all the subgroups of  $H$  conjugate under  $G$  to nonidentity subgroups of  $E$ ; the type is equal to the 2-rank, and the incidence is defined by inclusion. Then  $\mathcal{G}$  fulfills the axioms of a geometry (except for the strong connectedness, which must be checked separately),  $G$  acts flag-transitively on  $\mathcal{G}$ , and  $\mathcal{G}$  belongs to a string diagram

(i.e., without branches or loops) in which the residue of an element of type  $n$  is isomorphic to the projective geometry of the proper subgroups of  $E$ .

The following configurations give rise to  $T$ - and  $P$ -geometries; cf. [IS2]. In the first class of examples  $H$  is a suitable  $GF(2)G$ -module. For  $G \cong 3 \cdot \text{Sp}_4(2)$  or  $S_5$  it is the natural module of  $\Gamma L_3(4) > G$  (the hexacode); for  $G \cong M_{24}$  or  $\text{Aut}(M_{22})$  it is the Golay cocode; for  $G \cong 3 \cdot \text{Aut}(M_{22})$  it is the natural module of  $\Gamma U_6(2) > G$ ; for  $G \cong C_{01}$  or  $C_{02}$  it is  $\Lambda/2\Lambda$ , where  $\Lambda$  is the Leech lattice; finally, for  $G \cong He$  it is a 51-dimensional rational module, taken modulo 2 [MS]. Notice that the conjugates of  $E$  need not span the whole of  $H$ , say; in the case of  $G \cong C_{02}$  the conjugates of  $E$  span a submodule of codimension 1 in  $\Lambda/2\Lambda$ . In its turn, this submodule has a 22-dimensional quotient, which can also be taken as  $H$ .

For  $G \cong J_4$ ,  $BM$ , or  $M$  the role of  $H$  is played by  $G$  itself with the natural action by conjugation. In each of these cases  $G$  contains an involution  $\langle \tau \rangle$ , such that  $C = C_G(\tau)$  has the form  $C = Q.A$  where  $Q$  is an extraspecial group of order  $2^{1+m}$  for  $m = 12, 22$ , or  $24$ , respectively. Let  $\bar{Q} = Q/\langle \tau \rangle$ . Then the action of  $A$  (isomorphic, respectively, to  $3 \cdot \text{Aut}(M_{22})$ ,  $C_{02}$ , or  $C_{01}$ ) on  $\bar{Q}$  corresponds to a certain configuration from the previous paragraph. In particular, there is a subgroup  $\bar{E}$  in  $\bar{Q}$ , which gives rise to the geometry  $\mathcal{S}(A)$ . We take  $E$  to be the full preimage of  $\bar{E}$  in  $Q$ . In each case it is easy to see that  $N_G(E)$  induces on  $E$  the full linear group, and it is almost by the definition that the residue in  $\mathcal{S}(G, G, E)$  of the element  $\langle \tau \rangle$  coincides with  $\mathcal{S}(A, \bar{Q}, \bar{E})$ .

An additional class of examples can be constructed as  $i$ -covers. Let  $G \cong \text{Sp}_{2n}(2)$ ,  $\text{Aut}(M_{22})$ ,  $C_{02}$ , or  $BM$  and  $\mathcal{S} = \mathcal{S}(G)$  be the corresponding geometry. By  $\mathcal{S}(\text{Sp}_{2n}(2))$  we mean the classical  $C_n(2)$ -geometry. Consider in  $G$  a subgroup  $L$  isomorphic to  $\Omega_{2n}^-(2).2$ ,  $L_3(4).2$ ,  $U_6(2).2$ , or  $2 \cdot {}^2E_6(2).2$ , respectively. Then  $L/L' \cong Z_2$  (here  $L'$  is the derived group of  $L$ ), and there is a unique nontrivial 1-dimensional  $GF(3)L$ -module  $X$  whose kernel is  $L'$ . Let  $Y$  be the  $GF(3)G$ -module induced from  $X$  and  $\hat{G} \cong Y.G$  be a certain extension of  $G$  by  $Y$ . We consider the split extension in the first case and nonsplit extensions in the other cases. For  $L' \cong L_3(4)$ ,  $U_6(2)$ , or  $2 \cdot {}^2E_6(2)$  the 3-part of the Schur-multiplier of  $L'$  is of order 3. Moreover, every element from  $L - L'$  when extended to an automorphism of the nonsplit triple cover of  $L'$  inverts the center. Now by the Eckmann-Shapiro lemma (cf. Shapiro's lemma in [Bro]) in each of these three cases there exists a unique nonsplit extension  $\hat{G}$ . It can be shown that the amalgam of the rank  $i$  parabolic subgroups from  $G$  ( $i = 1$  for  $G \cong \text{Sp}_{2n}(2)$ , and  $i = 2$  otherwise) is embedded in  $\hat{G}$ . In the first case there is a trivial embedding into a complement to  $Y$ , and we consider another one. The embedded amalgam generates in  $\hat{G}$  a nonsplit extension  $3^{1/2} \cdot \text{Sp}_{2n}(2)$ ,  $3 \cdot \text{Aut}(M_{22})$ ,  $3^{23} \cdot C_{02}$ , or  $3^{4371} \cdot BM$ , respectively (where  $\begin{bmatrix} n \\ 2 \end{bmatrix}_2 = (2^n - 1)(2^n - 2)/6$ ). This extension corresponds to a 1-covering of  $\mathcal{S}(\text{Sp}_{2n}(2))$  by a flag-transitive  $T$ -geometry in the first case and to a 2-covering of  $\mathcal{S}$  by a  $P$ -geometry in the other three cases. These constructions were accomplished in [IS6, Shp3, IS7]. The symplectic series of  $T$ -geometries was also independently constructed by U. Meierfrankenfeld [Mei]. Notice that the exceptional  $C_3(2)$ -geometry for the group  $A_7$  does not have a 1-cover which is a  $T$ -geometry, as was checked in [IS3].

As indicated in column six of Table 1 on page 175 many of the  $T$ - and  $P$ -geometries contain subgeometries of  $T$ -,  $P$ -, or  $C_n(2)$ -type. These subgeometries play a crucial inductive role in the classification.

### 3. SIMPLE CONNECTEDNESS

Within the approach we choose, the following result constitutes a very important step of the classification.

**Theorem 3.1.** *The set of geometries in Table 1 with rank at least 3 is closed with respect to taking flag-transitive covers and quotients.*

Of course, the hard part of this statement is that every geometry in the table with rank at least 3, except  $\mathcal{G}(M_{22})$ , is simply connected (i.e., has no nontrivial coverings). This is equivalent to the fact that the automorphism group of the geometry coincides with the universal completion  $U(\mathcal{A})$  of the corresponding amalgam  $\mathcal{A}$  of maximal parabolic subgroups.

Since  $U(\mathcal{A})$  can be defined in terms of generators and relations, for its identification one can use the coset enumeration algorithm. This was implemented in [Hei] to check the simple connectedness of  $\mathcal{G}(M_{24})$ ,  $\mathcal{G}(He)$ ,  $\mathcal{G}(3^7 \cdot \text{Sp}_6(2))$ . Later an independent computer-free proof for the case of  $\mathcal{G}(M_{24})$  was found by the first author.

Another strategy goes back to [Ron1] and relies on analysis of cycles in the collinearity graph  $\Gamma$  of  $\mathcal{G}$ . The collinearity graph has  $\mathcal{G}^1$  as the set of vertices with two vertices adjacent if they are incident to a common element from  $\mathcal{G}^2$ . In many important cases it can be shown that a covering of  $\mathcal{G}$  induces a covering of its collinearity graph and that with respect to the induced covering all triangles are null-homotopic. In this case to prove the simple connectedness of  $\mathcal{G}$ , it is sufficient to show that  $\Gamma$  is *triangulable*, which means by definition that every cycle in  $\Gamma$  can be decomposed into a product of triangles.

Proceeding by induction on the rank and the number of elements, we can assume that all  $T$ - and  $P$ -subgeometries (as well as  $C_n(2)$ -subgeometries, if any) in  $\mathcal{G}$  are simply connected. Then each cycle of  $\Gamma$  which completely lies in a subgeometry is null-homotopic, and the analysis can be simplified considerably. This scheme was realized in [IS3] for  $\mathcal{G}(M_{23})$  and in [Shp3] for  $\mathcal{G}(C_{02})$ .

For larger geometries it turned out to be more convenient to consider a different graph  $\Sigma = \Sigma(\mathcal{G})$  (the intersection graph of subgeometries). The vertices of  $\Sigma$  are subgeometries, and two subgeometries are adjacent if they have "large" intersection. It was shown that every covering of  $\mathcal{G}$  induces a covering of  $\Sigma$  and that all triangles are null-homotopic with respect to the induced covering. After that it was proved that  $\Sigma$  is triangulable and hence  $\mathcal{G}$  is simply connected. This approach was realized for  $\mathcal{G}(C_{01})$ ,  $\mathcal{G}(J_4)$ ,  $\mathcal{G}(BM)$ , and  $\mathcal{G}(M)$  in [Ivn9, Ivn7, Ivn8, Ivn6], respectively.

For the remaining geometries the simple connectedness proof involves both combinatorial and group-theoretical arguments. In these cases  $O_3(G) \neq 1$ , and the homomorphism  $\phi : G \rightarrow \bar{G} = G/O_3(G)$  induces a morphism of  $\mathcal{G}(G)$  onto a geometry  $\mathcal{G}(\bar{G})$ , and the latter one is already known to be simply connected. We prove that for the amalgam  $\mathcal{A}$  of maximal parabolics the inequality  $|U(\mathcal{A})| \leq |G|$  holds [IS6, IS7, Shp3].

## 4. NATURAL REPRESENTATIONS

The geometries  $\mathcal{G}(G, H, E)$  from Section 2, where  $H$  is a  $GF(2)G$ -module, are constructed in their “natural representations”. By a *natural representation* of a  $T$ -,  $P$ -, or  $C_n(2)$ -geometry  $\mathcal{G}$  we understand its morphism  $\phi$  into the projective geometry of the nontrivial subspaces of a  $GF(2)$ -space, such that for  $x \in \mathcal{G}^i$  we have  $\dim(\phi(x)) = i$  and for  $j \leq i$  the restriction of  $\phi$  to the set of  $j$ -type elements from  $\text{res}_{\mathcal{G}}(x)$  is a bijection onto the set of  $j$ -dimensional subspaces of  $\phi(x)$ .

As all our geometries are of  $GF(2)$ -type (three points on a line), the natural representations of a fixed geometry  $\mathcal{G}$  (if they exist at all) are controlled by a particular one known as the *universal natural representation*. The corresponding  $GF(2)$ -vector space can be defined as the largest space spanned by vectors  $v_p$ ,  $p \in \mathcal{G}^1$ , such that  $v_a + v_b + v_c = 0$  whenever  $\{a, b, c\} = \text{res}_{\mathcal{G}}(\ell)^1$  for an element  $\ell \in \mathcal{G}^2$ . For  $\mathcal{G} = \mathcal{G}(G)$  this largest vector space  $UM(\mathcal{G})$  can be considered as a  $G$ -module and is called the *universal natural module*.

The above notions proved to be useful in the local characterization of the amalgams of maximal parabolics, where we need information on certain natural modules. In principle, the determination of the dimension of  $UM(\mathcal{G})$  is a problem of linear algebra over  $GF(2)$ . In particular, for the groups  $G \cong 3 \cdot S_6$  and  $S_5$  we can easily determine  $\dim(UM(\mathcal{G}(G)))$ . Another and much more impressive computation is due to B. McKay [McK], who established  $\dim(UM(\mathcal{G}(He))) = 52$  by considering on computer a system of 437,325 equations over 29,155 variables.

In all the other cases the proofs are computer-free [IS2, RSm2, Smi, IS4, IS5, Shp4]. Summarizing these results, we obtain the following.

**Theorem 4.1.** *The dimensions of the universal natural representations of  $T$ - and  $P$ -geometries are as given in column seven of Table 1.*

If, for the automorphism group  $G$ , we have  $O_3(G) \neq 1$ , then in Table 1 the dimension of the universal natural module is given in two summands: the commutant and the centralizer of  $O_3(G)$ , respectively.

Of particular interest are the arguments for the groups  $G \cong J_4, BM, 3^{4371} \cdot BM$ , and  $M$ . If in the above definition of  $UM(\mathcal{G})$  we do not assume that the elements  $v_p$  commute, we obtain the definition of the *universal natural group*  $UG(\mathcal{G})$ . If  $G$  is one of the above groups and  $\mathcal{G} = \mathcal{G}(G)$ , then  $G$  is a quotient of  $UG(\mathcal{G})$  (cf. Section 2). Let  $\xi : UG(\mathcal{G}) \rightarrow G$  be the corresponding homomorphism. It was shown that the restriction of  $\xi$  to a certain subgroup  $Q \leq UG(\mathcal{G})$  is an isomorphism and  $Q'$  contains  $v_p$  for some  $p \in \mathcal{G}^1$ . This readily implies that  $UG(\mathcal{G})$  has no abelian quotients, i.e.,  $UM(\mathcal{G})$  is trivial. The nonexistence of the natural representations of  $\mathcal{G}(M)$  answers a question posed in [Str].

**Conjecture 4.2.** *Let  $G \cong J_4, BM$ , or  $M$ . Then  $UG(\mathcal{G}(G))$  is isomorphic to  $J_4, 2 \cdot BM$ , or  $M$ .*

At the moment the conjecture is proved for  $G \cong BM$  and  $M$  [IPS].

## 5. LOCAL CHARACTERIZATION

On the stage of local characterization we prove that the amalgam  $\mathcal{A} = \{M_1, \dots, M_n\}$  of maximal parabolics corresponding to a flag-transitive action on a  $T$ - or  $P$ -geometry is isomorphic to that from a relevant known example.

Partial results on local characterization of  $T$ -geometries were proved in [Hei, Row1, Row2, Row3, Tim]. Compared with these papers, we use a somewhat different approach; in particular, we do not assume finiteness of parabolics. The approach was developed in [Shp1, Shp2] for  $P$ -geometries, and it works for  $T$ -geometries as well. To simplify the notation, let us consider only one of the two types of geometries, say,  $T$ -geometries.

Let  $\Delta$  be the graph on  $\mathcal{G}^n$  ( $n$  is the rank of  $\mathcal{G}$ ), in which two elements are adjacent exactly when they are incident to a common element of type  $n-1$  ( $\Delta$  is called the *derived graph* of  $\mathcal{G}$ ). For  $y \in \mathcal{G}^i$ ,  $i < n$ , let  $\Sigma(y)$  be the subgraph of  $\Delta$  induced by the elements (vertices) incident to  $y$ . Let  $\mathcal{O}$  be the set of the subgraphs  $\Sigma(y)$ . For  $y$  of type  $i < n-1$  the subgraph  $\Sigma(y)$  is naturally isomorphic to the derived graph of the residual  $T$ -geometry of rank  $n-i$ , related to  $y$ . If  $y \in \mathcal{G}^{n-1}$ , then  $\Sigma(y)$  is a 3-clique. It follows from the diagram of  $\mathcal{G}$  that the subgraphs  $\Sigma \in \mathcal{O}$  containing a particular vertex  $a$  (we denote this set of subgraphs by  $\mathcal{O}_a$ ) naturally form an  $(n-1)$ -dimensional projective space over  $GF(2)$ . The graph  $\Delta$  together with the set  $\mathcal{O}$  gives another realization of  $\mathcal{G}$ .

The group  $G$  acts naturally on  $\Delta$  preserving  $\mathcal{O}$  and having  $M = M_n$  as the stabilizer of a vertex  $a$ . Let  $K_s$  be the subgroup of  $M_n$  fixing all the vertices at a distance at most  $s$  from  $a$ . Consider the following condition.

(\*)  $K_{n-1}$  (where, as above,  $n$  is the rank of  $\mathcal{G}$ ) is a group of order at most 2.

When (\*) holds for all the residual  $T$ -geometries up to rank  $i < n$ , we can prove that  $K_i/K_{i+1}$  is an irreducible  $GF(2)$ -module for the quotient  $L_n(2)$  of  $M$  (this quotient is the action of  $M$  on  $\mathcal{O}_a$ ), of dimension 0, 1, or  $\binom{n}{i}$ . The proof goes as follows. Every vertex at distance  $i$  from  $a$  is covered by a subgraph  $\Sigma(y) \in \mathcal{O}_a$  for a  $y$  of type  $n-i$ . By (\*),  $K_i$  induces on  $\Sigma(y)$  the action of order at most 2. If the action is trivial,  $K_i/K_{i+1}$  is trivial as well. Otherwise, in the dual of  $K_i/K_{i+1}$  there is an orbit indexed by the  $y$ 's, which are simply all the elements of type  $i$  in the projective space  $\text{res}_{\mathcal{G}}(a) \cong \mathcal{O}_a$ . We can check that the vectors from this orbit (if not all equal) possess some natural 3-term linear relations, which immediately leads to the identification of  $K_i/K_{i+1}$ .

If all the proper residual  $T$ -geometries have the property (\*), then some further simple arguments show  $K_{n+1} = 1$  and enable us to determine the structure of  $K_n$ . As soon as all the chief factors of  $M$  are known, reconstruction of the possible amalgams  $\mathcal{A}$  is only a matter of technique.

Since the condition (\*) holds for all the known  $T$ -geometries, except for the terminal (in the tree of  $T$ -geometries) geometry  $\mathcal{G}(M)$ , we can inductively reconstruct all the amalgams corresponding to the existing  $T$ -geometries. It remains to prove that the  $T$ -geometry  $\mathcal{G}(M)$  has no further extension. For such an extension we can determine, as above, four first sections  $K_i/K_{i+1}$ . However, we cannot immediately conclude that  $K_{n+1} = 1$  (here  $n = 6$ ). Let

us consider the parabolic  $M_1$  which is the stabilizer in  $G$  of an element  $u \in \mathcal{E}^1$ . Let  $N$  be the kernel of  $M_1$  acting on  $\text{res}_{\mathcal{E}}(u)$  (equivalently, on  $\Sigma(u)$ ) and  $N_1$  be the kernel of  $N$  acting on the set of elements from  $\mathcal{E}^1$  which are incident with  $u$  to a common element from  $\mathcal{E}^2$ . Then  $M_1/N$  is a flag-transitive action on  $\text{res}_{\mathcal{E}}(u)$  and is known to be isomorphic to the Monster group  $M$  by the inductive hypothesis. We prove that  $N/N_1$  is a  $GF(2)$ -space and its dual realizes a natural representation of  $\text{res}(u)$ . By Theorem 4.1 this module must be trivial, i.e.,  $N = N_1$ . Then we prove, using (\*) for the residues up to rank 4, that  $N$  stabilizes all the vertices at distance  $\leq 4$  from  $\Sigma(u)$ , which means that  $N \leq K_4$ . However, the 2-part of the order of  $M_n/K_4$  is greater than that of  $M_1/N \cong M$ , unless the section  $K_3/K_4$  is already trivial or 1-dimensional. In the latter case we obtain that  $K_4 = 1$  and eventually establish a contradiction.

## 6. APPLICATIONS

Let us discuss some consequences of the Main Theorem and its proof.

**Maximal parabolic geometries.** The simple connectedness of the minimal parabolic geometries of the groups  $M_{24}$ ,  $He$ ,  $Co_1$ ,  $Co_2$ ,  $J_4$ ,  $BM$ , and  $M$  implies the simple connectedness of the corresponding maximal parabolic geometries (proved in [Ron2] and [Seg1] for the cases  $M_{24}$  and  $Co_1$ , respectively). The fact that the universal covering of  $\mathcal{E}(M_{22})$  is isomorphic to  $\mathcal{E}(3 \cdot M_{22})$  implies the simple connectedness of the maximal parabolic geometry for  $M_{22}$ .

**Graphs.** First examples of  $P$ -geometries were constructed in terms of the corresponding derived graphs [Ivn1]. These graphs satisfy the following.

**Hypothesis 6.1.**  $\Gamma$  is a graph whose girth (the length of the shortest cycle) is 5.  $G = \text{Aut}(\Gamma)$  acts vertex- and edge-transitively on  $\Gamma$ . For a vertex  $x$  the action  $G(x)^{\Gamma(x)}$  of the vertex stabilizer  $G(x)$  on the set  $\Gamma(x)$  of vertices adjacent to  $x$  is doubly transitive without regular normal subgroups. The kernel  $G_1(x)$  of this action is nontrivial.

In [Ivn2] and [Ivn3] the classification problem of graphs satisfying Hypothesis 6.1 was reduced to the classification of the flag-transitive  $P$ -geometries. Now we can formulate the final result (cf. [Ivn11]).

**Theorem 6.2.** Let  $\Gamma$  be a graph satisfying Hypothesis 6.1. Then  $\Gamma$  is either the derived graph of one of the following  $P$ -geometries:  $\mathcal{E}(M_{22})$ ,  $\mathcal{E}(3 \cdot M_{22})$ ,  $\mathcal{E}(Co_2)$ ,  $\mathcal{E}(3^{23} \cdot Co_2)$ ,  $\mathcal{E}(J_4)$ ,  $\mathcal{E}(BM)$ , and  $\mathcal{E}(3^{4371} \cdot BM)$ , or it is a graph of valency 31 which is related to the derived graph of  $\mathcal{E}(J_4)$ .

In the derived graph of  $\mathcal{E}(M_{23})$  the condition  $G_1(x) \neq 1$  fails.

The graphs in Theorem 6.2 are extremal by the order of the vertex stabilizer in the general class of graphs with doubly transitive action  $G(x)^{\Gamma(x)}$  [Tro].

**Uniqueness of certain sporadics.** Construction of the geometries  $\mathcal{E}(G)$  for  $G \cong J_4$ ,  $BM$ , and  $M$  relies exclusively on the structure of the involution centralizer  $C = C_G(\tau)$  and on certain information on fusion in  $G$  of involutions from  $O_2(C_G(\tau))$ . The characterization of the geometries  $\mathcal{E}(G)$  implies the following.

**Theorem 6.3.** *Let  $G$  be a nonabelian simple group containing an involution  $\tau$  such that  $C_G(\tau)$  is of the shape  $2^{1+12}.3 \cdot \text{Aut}(M_{22})$ ,  $2^{1+22}.Co_2$ , or  $2^{1+24}.Co_1$ . Suppose that  $C_G(O_2(C)) \leq O_2(C)$  and that  $\tau^G \cap O_2(C) \neq \{\tau\}$ . Then  $G$  is uniquely determined and is isomorphic to  $J_4$ ,  $BM$ , or  $M$ , respectively.*

For the original uniqueness proofs for the groups  $J_4$ ,  $BM$ , and  $M$  see [Nor1, AS, LS, Seg2, Tho, Nor2, GMS].

**Generators and relations.** The classification of  $T$ - and  $P$ -geometries enables us to obtain a characterization of certain sporadic simple groups stronger than the characterization by the centralizer of an involution. The groups are proved to coincide with the universal completions of certain of their subamalgams. This provides us with presentations of the groups involved (the *geometric presentations* [Ivn4]). In the case of  $J_4$  the geometric presentation was proved in [Ivn7] to be equivalent to a nice presentation conjectured by G. Stroth and R. Weiss in [SW].

In the case of  $BM$  and  $M$  the result establishes the correctness of the so-called  $Y$ -presentations for these groups. The  $Y$ -presentations (cf. [Atlas, CNS, Nor3]) describe groups as specific factor groups of Coxeter groups with diagrams having three arms originating in a common node. The most famous is the  $Y_{555}$ -diagram with three arms of five edges each. The nodes on the arms are denoted by  $a, b_i, c_i, d_i, e_i$ , and  $f_i$  for  $i=1, 2$ , and  $3$ . After the announcement of the geometric presentation of  $M$  in [Ivn6], S. Norton [Nor4] proved its equivalence to the corresponding  $Y$ -presentation. This resulted in the proof of the following theorem conjectured by J. Conway [Con].

**Theorem 6.4.** *The Coxeter group corresponding to the  $Y_{555}$ -diagram subjected to a single additional relation  $(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1$  is isomorphic to the wreath product  $M \text{ wr } 2$  of the Monster group and a group of order 2 (this wreath product is known as the Bimonster).*

The correctness of the  $Y$ -presentation for  $BM$  is proved in [Ivn10].

**Representations and cohomologies.** Within the classification of  $T$ - and  $P$ -geometries and their natural representations, considerable information on linear representations and nonsplit extensions of sporadic groups was obtained. We formulate here only one result of this type which can be deduced from [IS7].

**Theorem 6.5.** *Let  $K$  be a field whose characteristic is not 2. Then  $BM$  has a unique representation over  $K$  of dimension 4371. If  $\text{char}(K) \neq 3$ , then the extension of  $BM$  by the corresponding module always splits, and for  $K = GF(3)$  there is a unique nonsplit extension.*

**Constructions.** Since the sporadic groups involved turned out to coincide with the universal completions of certain of their subamalgams, there is a possibility of producing an independent construction of these groups. Such a construction is now in progress for  $J_4$  where the corresponding amalgam is realized in  $GL_{1333}(C)$  [IM]. A possibility to use the geometric characterization of the Monster group in order to simplify its construction is discussed in [Ivn12].

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