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EXCEPTIONAL SURGERY ON KNOTS

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ABSTRACT. Let M be an irreducible, compact, connected, orientable 3-manifold whose boundary is a torus. We show that if M is hyperbolic, then it admits at most six finite/cyclic fillings of maximal distance 5. Further, the distance of a finite/cyclic filling to a cyclic filling is at most 2. If M has a non-boundary-parallel, incompressible torus and is not a generalized 1-iterated torus knot complement, then there are at most three finite/cyclic fillings of maximal distance 1. Further, if M has a non-boundary-parallel, incompressible torus and is not a generalized 1- or 2-iterated torus knot complement and if M admits a cyclic filling of odd order, then M does not admit any other finite/cyclic filling. Relations between finite/cyclic fillings and other exceptional fillings are also discussed.

Let M be a compact, connected, orientable 3-manifold whose boundary is a torus. M is called *irreducible* if any embedded 2-sphere in M bounds an embedded 3-ball. Otherwise, M is called *reducible.* A *slope* on ∂M is a ∂M-isotopy class r of essential, simple closed curves. A slope r is called a *boundary* slope if there is a properly embedded, orientable, incompressible, ∂-incompressible surface F in M such that ∂F is a nonempty set of parallel simple closed curves on ∂M of slope r. The *distance* between two slopes r_1 and r_2, denoted by $\Delta(r_1, r_2)$, is the minimal geometric intersection number between r_1 and r_2. As ∂M is a torus, $\Delta(r_1, r_2)$ may be calculated as the absolute value of the algebraic intersection number of the homology classes carried by r_1 and r_2.

Given a slope r on ∂M, a well-defined closed 3-manifold $M(r)$ can be constructed by attaching a solid torus to M via a homeomorphism which identifies a meridian curve of the solid torus to a representative curve for r. $M(r)$ is called the *r-filling* of M. Given a knot K in a closed, connected, orientable 3-manifold W, with tubular neighbourhood $N(K)$, exterior $M = W - \operatorname{int}(N(K))$, and slope r on ∂M, $M(r)$ is also referred to as the *r-surgery* of W along K.

A fundamental result of Wallace [Wa] and Lickorish [Li] states that each closed, orientable 3-manifold results from surgery on some link in the 3-sphere. Thus a natural approach to 3-manifold topology is to analyze to what extent various aspects of the topology of a manifold M, as above, are inherited by the manifolds $M(r)$. For instance, one could try to determine when a closed essential surface in M becomes inessential in some $M(r)$ or when an irre-

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ducible M could produce a reducible $M(r)$. An excellent survey of this topic may be found in [Go1]. Another example of some importance arises when M is a hyperbolic manifold, i.e., $\text{int}(M)$ admits a complete hyperbolic metric of finite volume. Thurston [Th] has shown that in this situation, all but finitely many of the manifolds $M(r)$ are hyperbolic. The nonhyperbolic slopes include those whose fillings are:

(a) manifolds with finite or cyclic fundamental groups,
(b) manifolds which are reducible,
(c) manifolds which are Seifert-fibred spaces,
(d) manifolds which admit an incompressible torus.

Thurston's geometrication conjecture [Th] predicts that the remaining slopes yield fillings which are hyperbolic.

A basic problem then is to describe the set of *exceptional* slopes on a torally bounded manifold M, i.e., those slopes producing nongeneric fillings. An appropriate description should include an upper bound on the number of such slopes as well as a qualitative measure of their relative positions determined by bounds on their mutual distances. We call a slope r on ∂M *cyclic* if $M(r)$ has cyclic fundamental group; and similarly we shall refer to finite slopes, reducible slopes, Seifert slopes, and essential torus slopes. It is a remarkable result, the *cyclic surgery theorem* [CGLS], that if M is not a Seifert-fibred space, then all cyclic slopes on ∂M have mutual distance no larger than 1, and consequently there are at most three such fillings. Gordon and Luecke obtained similar estimates for the set of reducible slopes [GLu1] and also examined essential torus slopes [Go2, GLu2]. In another direction, Bleiler and Hodgson [BH] refined results of Gromov and Thurston to obtain restrictions on the slopes on a hyperbolic M which do not produce manifolds admitting metrics of strictly negative sectional curvature. We consider the problem of determining the set of finite slopes on M here. Henceforth, we shall use finite/cyclic to mean either infinite cyclic or finite. Standard arguments show that we may take M to be irreducible, and we shall assume this below. Then according to [Th], M belongs to one of the following three mutually exclusive categories:

(I) M is a Seifert-fibred space admitting no essential, i.e., incompressible and non-∂-parallel torus.
(II) M is a hyperbolic manifold.
(III) M contains an essential torus.

It turns out to be convenient to consider these three cases separately. In case (I) it is well known that one can completely classify the finite/cyclic fillings of M. Considering the torus knots for instance, one sees that there exist infinitely many knots whose exteriors are of type (I), each of which admit an infinity of distinct finite (cyclic or noncyclic) surgery slopes. Our contributions deal with the cases (II) and (III). For the former we obtain

Theorem A. *Let K be a knot in a closed, connected, orientable, 3-manifold W, such that the interior of $M = W - \text{int}N(K)$ admits a complete hyperbolic structure of finite volume.*

(1) *There are at most six finite/cyclic surgeries on K, and $\Delta(r_1, r_2) \leq 5$ for any two finite/cyclic surgery slopes r_1 and r_2 of K.*

(2) *If* r_1 *is a finite/cyclic surgery slope of* K *and* r_2 *is a cyclic surgery slope of* K, *then* $\Delta(r_1, r_2) \leq 2$.

The known realizable maximal number of finite/cyclic surgeries on a knot as in Theorem A and their maximal mutual distance is 5 and 3 [We]. This example may also be used to show that Theorem A(2) is sharp.

To discuss case (III), we introduce the following notion. A compact, connected, orientable, 3-manifold M, with boundary a torus, is called a *generalized n-iterated torus knot exterior* if M can be decomposed along disjoint, essential tori into a union of n cabled spaces (in the sense of [GLi]) and a Seifert-fibred space which has a Seifert fibration over the 2-disc with exactly two exceptional fibres.

Theorem B. *Let* K *be a knot in a closed, connected, orientable, 3-manifold* W, *such that* $M = W - \text{int}N(K)$ *is irreducible and contains an essential torus.*

(1) *If* M *is not a generalized* 1*-iterated torus knot exterior, then* $\Delta(r_1, r_2) \leq 1$ *for any two finite/cyclic surgery slopes* r_1 *and* r_2 *of* K. *In particular, there are at most three finite/cyclic surgeries on* K.

(2) *If* M *is not a generalized* 1- *or* 2*-iterated torus knot exterior and if* K *admits a cyclic surgery of odd order, then* K *does not admit any other finite/cyclic surgery.*

Finite/cyclic fillings on a generalized 1- or 2-iterated torus knot exterior M can be completely described. This is essentially done in [BH, §2], where it is shown that if M is not a union of the twisted I-bundle over the Klein bottle and a cabled space, then there are no more than six finite/cyclic fillings of maximal mutual distance 5 (realized on the exterior of the $(11, 2)$-cable over the $(2, 3)$-torus knot in S^3). In particular, it is proved that an iterated torus knot in S^3, admitting a nontrivial finite surgery, must be a cable over a torus knot. A complete list of all finite surgeries on cabled knots over torus knots in S^3 is given in §2 of that paper.

It is shown in [BZ1, Example 10.6] that Theorem B(1) is sharp. We also note that as the finite/cyclic fillings on generalized 1- or 2-iterated torus knot exteriors are readily determined, Theorem B(2) completes the classification of finite/cyclic surgeries on knots in manifolds of odd-order, cyclic fundamental group whose exteriors contain an essential torus.

Consider now surgery on knots in the 3-sphere S^3. As is usual, slopes for a knot in S^3 are parameterized by $\mathbf{Q} \cup \{\frac{1}{0}\}$ through the use of the standard meridian-longitude coordinates [R]. In S^3 only the trivial knot admits a $\mathbf{Z}$-surgery [Ga3].

Corollary C. *Let* $K \subset S^3$ *be a hyperbolic knot.*

(1) *There are at most six finite surgeries on* K, *and* $\Delta(r_1, r_2) \leq 5$ *for any two finite surgery slopes* r_1 *and* r_2 *of* K.

(2) *If* r_1 *is a finite surgery slope of* K *and* r_2 *is a cyclic surgery slope of* K, *then* $\Delta(r_1, r_2) \leq 2$. *In particular, if* m/n *is a finite surgery slope of* K, *then* $|n| \leq 2$.

It is shown in [BH] that the $(-2, 3, 7)$-pretzel knot admits at least four finite surgeries of maximal mutual distance 2. We prove in [BZ1, Example 10.1] that this knot has no other finite slopes. This example exhibits the known maximal number of finite surgeries on a hyperbolic knot in S^3.

Corollary D. *Let $K \subset S^3$ be a satellite knot. If K admits a nontrivial finite surgery, then K is a cabled knot over a torus knot.*

It follows from our previous remarks that Corollary D classifies finite surgeries on satellite knots in S^3.

In their recent work [BH], Bleiler and Hodgson obtained, using a completely different approach, the number 24 and the distance 23 for finite/cyclic surgery on a knot as in Theorem A and the number 8 and the distance 5 for finite/cyclic surgery on a knot as in Theorem B.

It is a classic result [Mi] that a finite group, which is the fundamental group of a 3-manifold, must belong to one of the following types: C-type, cyclic groups; D-type, dihedral-type groups; T-type, tetrahedral-type groups; O-type, octahedral-type groups; I-type, icosahedral-type groups; and Q-type, quaternionic-type groups. It is shown in [BZ1] that more precise information on finite/cyclic surgeries of a given type may be obtained. For knots in S^3 these yield the following result.

Proposition E. *Let $K \subset S^3$ be a hyperbolic knot.*

(1) (i) *Any* D*-type slope of K must be an integral slope.*

(ii) *There is at most one* D*-type finite surgery on K.*

(iii) *If there is a* D*-type surgery on K, then there is no even-order cyclic surgery on K.*

(iv) *If there is a* D*-type surgery α on K, then there is at most one nontrivial cyclic surgery on K; and if there is one, β, say, then α and β are consecutive integers.*

(2) *There are at most two* T*-type slopes on K; and if two, one is integral, the other has denominator 2, and their distance is 3.*

(3) (i) *Any* O*-type slope of K must be an integral slope.*

(ii) *There are at most two* O*-type slopes on K; and if two, their distance is 4.*

(iii) *If there is an* O*-type surgery on K, then there is no even-order cyclic surgery on K.*

(iv) *If there is an* O*-type surgery α on K, then there is at most one nontrivial cyclic surgery on K; and if there is one, β, say, then α and β are consecutive integers.*

(v) *If there is an* O*-type surgery slope and a* D*-type surgery slope on K, then they are consecutive even integers.*

Proposition E may be complemented by various examples of hyperbolic knots in S^3 which admit O-type, or I-type, or D-type surgery [BH]. In [BZ1] we produce a few more examples. Notably, [BZ1, Example 10.2] provides a hyperbolic knot in S^3 which admits a D-type surgery and an odd-order nontrivial cyclic surgery, and [BZ1, Example 10.4] provides a hyperbolic knot in S^3 which admits a T-type surgery and a nontrivial cyclic surgery.

Based on the results obtained in [BZ1] and known examples, we raise the following problem.

Finite/cyclic surgery problem. (I) Let K be a knot in a connected, closed, orientable 3-manifold W such that $M = W - \text{int}(N(K))$ is a hyperbolic manifold.

Show that there are at most five finite/cyclic surgeries on K and that the distance between any two finite/cyclic surgery slopes is at most 3.

(II) Let $K \subset S^3$ be a hyperbolic knot. Show that:

(1) there are at most four finite surgeries on K;
(2) the nontrivial finite surgery slopes on K form a set of consecutive integers;
(3) the distance between any two finite surgery slopes of K is at most 2;
(4) there is at most one finite surgery on K with an even integral slope.

Evidence supporting a positive solution to this problem may be found in [BZ1]. For instance, if the minimal norm [CGLS, Chapter 1] amongst all nontrivial elements of $H_1(\partial M)$ is greater than 24 (16 for knots whose exteriors have no 2-torsion in their homology), then the methods of [BZ1] show that for a knot K as in Theorem A there are at most four finite/cyclic surgeries on K of maximal mutual distance no more than 2. These methods may also be used to solve the finite/cyclic surgery problem for various families of knots, such as those with 2-bridges [Ta].

Proofs of the results listed above may be found in [BZ1]. The techniques used there are based on the work on M. Culler, C. M. Gordon, J. Luecke, and P. Shalen [CS, CGLS]. In particular, we derive results on the norm defined in Chapter 1 of [CGLS] which, when combined with Chapter 2 of that paper, yields Theorem A. Theorem B is proven by considering the torus decomposition of M and then applying Theorem A and results from [Gal, Ga2, Sch, CGLS] to determine what the pieces of this decomposition are, under the added assumption that there are two finite/cyclic slopes such that either (i) they are of distance greater than 1 apart, or (ii) one of them is a cyclic slope of odd order.

Our study of exceptional fillings is continued in [BZ2].

Theorem F. *Let M be a compact, connected, orientable, irreducible 3-manifold with ∂M a torus. Assume that M is not an atoroidal Seifert-fibred manifold. Fix slopes r_1 and r_2 on ∂M and suppose that $M(r_1)$ is a reducible manifold.*

(1) If $M(r_2)$ has a cyclic fundamental group, then $\Delta(r_1, r_2) \leq 1$.

(2) If M is hyperbolic and r_2 is a finite slope, then $\Delta(r_1, r_2) \leq 5$ unless $M(r_1) = \mathbf{R}P^3 \# \mathbf{R}P^3$ and $\pi_1(M(r_2))$ is a D-type group or a Q-type group.

(3) If M contains an essential torus and r_2 is a finite slope, then $\Delta(r_1, r_2) \leq 1$ unless M is a cable on the twisted I-bundle over the Klein bottle or a cable on a hyperbolic manifold for which the inequality of part (2) does not hold.

We also obtain a new proof of the following result of Gordon and Luecke.

Theorem G ([GLu1]). *Let M be a compact, connected, orientable, irreducible 3-manifold with ∂M a torus. If $M(r_i)$ is a reducible manifold, for $i = 1, 2$, then $\Delta(r_1, r_2) \leq 1$.*

Combining the last two results with the cyclic surgery theorem, we obtain

Corollary H. *Let M be a compact, connected, orientable, irreducible 3-manifold with ∂M a torus. Suppose that M is not an atoroidal Seifert-fibred space. If for $i = 1, 2$, $M(r_i)$ is either a reducible manifold or a manifold with cyclic fundamental group, then $\Delta(r_1, r_2) \leq 1$. Consequently there are at most three cyclic/reducible fillings on M.*

Suppose now that r_1 is a slope on ∂M such that $M(r_1)$ has the fundamental group of a Seifert-fibred manifold W and that r_2 is a finite/cyclic slope. Theorems A and B provide upper bounds for $\Delta(r_1, r_2)$ when W admits a Seifert-fibration with base orbifold the 2-sphere having no more than two exceptional fibres or three such fibres if their indices form a platonic triple. Our next theorem deals with most of the remaining cases.

Theorem I. *Let M be a compact, connected, orientable, 3-manifold with ∂M a torus. Suppose further that M is neither Seifert-fibred nor a cable on a Seifert-fibred manifold. Let r_1 be a slope on ∂M such that $M(r_1)$ has the fundamental group of a Seifert-fibred space which admits no Seifert fibration having base orbifold the 2-sphere with exactly three exceptional fibres. Then*

(1) $\Delta(r_1, r_2) \leq 1$ *if $M(r_2)$ has a cyclic fundamental group*;

(2) $\Delta(r_1, r_2) \leq 5$ *if $M(r_2)$ has a finite fundamental group unless $M(r_1)$ is either $\mathbf{R}P^3 \# \mathbf{R}P^3$ or a union of two copies of the twisted I-bundle over the Klein bottle, and $\pi_1(M(r_2))$ is a D-type group or a Q-type group.*

Applying this result to knots in S^3, we obtain

Corollary J. *Let M be the exterior of a knot K in S^3 and r a slope on ∂M such that $M(r)$ has the fundamental group of a Seifert-fibred space.*

(1) *If $M(r)$ has the fundamental group of a Seifert-fibred space which is Haken, then r is an integral slope.*

(2) *If K is a satellite knot which is not cabled exactly once, then r is an integral slope.*

Finally we return to the question of quantifying nonhyperbolic slopes on a hyperbolic manifold M. Call a slope on ∂M *big Seifert*, or bS for short, if the associated filling yields a Seifert-fibred manifold whose base orbifold is not a 2-sphere with fewer than four cone points. Using the results above as well as results from [GLi] and [Go2], we may prove the following.

Corollary K. *Let M be a compact, connected, orientable, hyperbolic 3-manifold with ∂M a torus. Let r and s be two slopes that are contained in the set of all reducible/cyclic/finite/bS slopes. If neither $M(r)$ not $M(s)$ is $\mathbf{R}P^3 \# \mathbf{R}P^3$ or a union of two copies of the twisted I-bundle over the Klein bottle, then $\Delta(r, s) \leq 5$. Hence there are at most eight such slopes.*

The reader will find further discussion and results on exceptional slopes in [BZ2].

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References

[BH] S. Bleiler and C. Hodgson, *Spherical space forms and Dehn fillings*, preprint.

[BZ1] S. Boyer and X. Zhang, *Finite Dehn surgery on knots*, preprint.

[BZ2] ——, *The semi-norm and Dehn filling*, preprint.

[CGLS] M. Culler, C. M. Gordon, J. Luecke, and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), 237–300.

[Ga1] D. Gagai, *Surgery on knots in solid tori*, Topology **28** (1989), 1–6.

[Ga2] ——, *1-bridge braids in solid tori*, Topology Appl. **37** (1990), 221–235.

[Ga3] ——, *Foliations and the topology of 3-manifolds*. III, J. Differential Geom. **26** (1987), 479–536.

[Go1] C. M. Gordon, *Dehn surgery on knots*, Proceedings of the International Congress of Mathematians, Kyoto, 1990, The Mathematical Society of Japan, Tokyo, Japan.

[Go2] ——, *Boundary slopes and punctured tori in 3-manifolds*, preprint.

[GLi] C. M. Gordon and R. A. Litherland, *Incompressible planar surfaces in 3-manifolds*, Topology Appl. **18** (1984), 121–144.

[GLu1] C. M. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, preprint.

[GLu2] ——, *Address*, John Luecke, 1993 Georgia International Topology Conference, University of Georgia at Athens, 1–13 August 1993.

[Li] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) **76** (1962), 531–540.

[Mi] J. Milnor, *Groups which act on S^n without fixed points*, Amer. J. Math. **79** (1957), 623–631.

[R] D. Rolfsen, *Knots and links*, Publish or Perish, Cambridge, MA, 1979.

[Sch] M. Scharlemann, *Producing reducible 3-manifolds by surgery on a knot*, Topology **29** (1990), 481–500.

[Ta] D. Tanguay, *Chirurgie finie et noeuds rationnels*, Doctoral dissertation, Université du Québec à Montréal, 1994.

[Th] W. Thurston, *Three-dimensional manifolds, Kleinian groups, and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), 357–381.

[Wa] A. H. Wallace, *Modifications and cobounding manifolds*, Canad. J. Math. **12** (1960), 503–528.

[We] J. Weeks, *Hyperbolic structures on three-manifolds*, Ph.D. thesis, Princeton University, 1985.

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