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ZETA FUNCTIONS DO NOT DETERMINE CLASS NUMBERS

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ABSTRACT. We show that two number fields with the same zeta function, and even with isomorphic adele rings, do not necessarily have the same class number.

INTRODUCTION

The Dedekind zeta function of a number field encodes many of its arithmetic invariants, including its degree, discriminant, number of roots of unity, number of real and complex embeddings, and for every rational prime the list of residue degrees of the extension primes. The zeta function also determines the product of the class number and the regulator. See [3, 4] for proofs.

It has been an open problem whether or not the zeta function of a number field determines its class number.

Theorem. *There exist two number fields with the same zeta function and different class numbers.*

Two number fields with the same zeta function are said to be *arithmetically equivalent.* Let H and H' be subgroups of the Galois group G of a Galois extension N of $\mathbb{Q}$, and denote by 1_H^G the character of G induced by the trivial character of H. Then the invariant fields $K = N^H$ and $K' = N^{H'}$ are arithmetically equivalent if and only if $1_H^G = 1_{H'}^G$. If $1_H^G = 1_{H'}^G$ and p is a prime not dividing the order of G, then the p-parts of the class groups of K and K' are isomorphic [4].

Explicit examples of non-isomorphic arithmetically equivalent fields are the fields $K = \mathbb{Q}(\sqrt[8]{a})$ and $K' = \mathbb{Q}(\sqrt[8]{16a})$, where a is an integer for which none of the numbers a, $-a$, $2a$, $-2a$ is a square (see [4]). The Galois closure of K and K' has degree 32, so the class number quotient h/h' is a power of 2.

NUMERICAL EVIDENCE

According to our computations with the system for computational number theory Pari/GP (version 1.38, 1993) by Henri Cohen [1, Appendix A], the values $a = -15$, -31, -33, -63, 65, 66 give pairs of fields K, K' with $h/h' = 2$. For $a = -65$, -66 we get $h'/h = 2$. Wieb Bosma has verified this using the Magma/KANT system in Sydney.

Of course this does not qualify as a proof of the theorem. Perhaps rodents in the bowels of the computer center are chewing on wires and altering data.

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More disconcerting is the fact that the correctness of the algorithms we used is based on unproven hypotheses; see [1].

Sketch of proof

Take $a=-15$, which gives arithmetically equivalent fields $K=\mathbb{Q}(\alpha)$ with $\alpha^8=-15$ and $K'=\mathbb{Q}(\beta)$ with $\beta^8=-240$.

We define the groups of units $U_0=\langle -1, \frac{\alpha+1}{\alpha-1}, \frac{\alpha^2+\alpha+2}{\alpha+1}, \frac{\alpha^2-\alpha+2}{-\alpha+1}\rangle$ and $U_0'=\langle -1, \frac{\beta^6+2\beta^4-4\beta^2-56}{16}, \frac{\beta^7-2\beta^6+2\beta^5-4\beta^3+8\beta^2-8\beta+64}{64}, \frac{\beta^7-2\beta^5+4\beta^4-4\beta^3-32\beta^2+8\beta-16}{64}\rangle$. The fractions were listed by Pari as fundamental units. We only use that they are units, which can be checked with a straightforward computation.

The regulators of U_0 and U_0' are computed to be $R_0\approx 66.316$ and $R_0'\approx 132.633$. Universal regulator bounds from [2] show that the regulators R and R' of the full unit groups U and U' are at least 0.296. It follows that $i=[U:U_0]$ and $i'=[U':U_0']$ are at most $\frac{133}{0.296}<500$. By [3] we have $hR=h'R'$, so $R_0'/R_0=i'/i\cdot h/h'$. If $h/h'\in\mathbb{Z}$, then it follows that the denominator of the rational number R_0'/R_0 is at most 500. If $h/h'\notin\mathbb{Z}$, then $h'/h\in\mathbb{Z}$, so the denominator divides $ih'/h=i'R_0/R_0'<500$. Our approximations of R_0 and R_0' show that $|2-R_0'/R_0|<10^{-3}$, and we deduce that $R_0'/R_0=2$.

For a unit $u\in U'$ and a prime $\mathfrak{p}$ of K', define $q(u,\mathfrak{p})\in\mathbb{F}_2$ by letting $(-1)^{q(u,\mathfrak{p})}$ be the quadratic residue of u modulo $\mathfrak{p}$. Letting u range over the four generators of U_0' and $\mathfrak{p}$ over the four prime ideals $(3,\beta)$, $(19,\beta-8)$, $(23,\beta-9)$, $(47,\beta-16)$, one can check that the (4×4)-matrix $(q(u,\mathfrak{p}))_{u,\mathfrak{p}}$ over $\mathbb{F}_2$ is non-singular. It follows that $U_0'\cap U'^2=U_0'^2$ and that i' is odd.

We now know that $h/h'=R_0'/R_0\cdot i/i'=2i/i'$ and that i' is odd. Counting factors 2, we see that the 2-power h/h' is at least 2. This proves the theorem.

With four suitably chosen primes of K one can also show that i is odd and that $h/h'=2$.

Adele rings

For each integer a and each odd prime p the $\mathbb{Q}_p$-algebras $\mathbb{Q}_p[X]/(X^8-a)$ and $\mathbb{Q}_p[X]/(X^8-16a)$ are isomorphic. If $a\equiv -1$ modulo 32, then this also holds for $p=2$, so the fields $K=\mathbb{Q}(\sqrt[8]{a})$ and $K'=\mathbb{Q}(\sqrt[8]{16a})$ have isomorphic adele rings. For $a=-33$ computations with Pari indicate that the class numbers are different, and this can be verified by imitating the argument above. It follows that the class number of a number field is not determined by the isomorphism class of its adele ring.

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