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A GENERAL DECOMPOSITION THEORY FOR RANDOM CASCADES

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ABSTRACT. This announcement describes a probabilistic approach to cascades which, in addition to providing an entirely probabilistic proof of the Kahane-Peyrière theorem for independent cascades, readily applies to general dependent cascades. Moreover, this unifies various seemingly disparate cascade decompositions, including Kahane's T-martingale decomposition and dimension disintegration.

1. BRIEF HISTORY OF THE PROBLEM

A theory of *positive T-martingales* was introduced in [K3] as the general framework for independent multiplicative cascades and random coverings. This includes spatial distributions of interest in both probability theory and in the physical sciences, e.g. [CCD, CK, DE, DG, DF, DM, F, GW, MS, MW, TLS, PW, S]. For basic definitions, let T be a locally compact metric space with Borel sigmafield $\mathscr{B}$, and let $(\Omega, \mathscr{F}, P)$ be a probability space together with an increasing sequence $\mathscr{F}_n$, $n = 1, 2, \ldots$, of sub-sigmafields of $\mathscr{F}$. A *positive T-martingale* is a sequence $\{Q_n\}$ of non-negative functions on $T \times \Omega$ such that: (i) for each $t \in T$, $\{Q_n(t, \cdot) : n = 1, 2, \ldots\}$ is a martingale adapted to $\mathscr{F}_n$, $n = 1, 2, \ldots$; (ii) for P-a.s. $\omega \in \Omega$, $\{Q_n(\cdot, \omega) : n = 1, 2, \ldots\}$ is a sequence of Borel measurable nonnegative real-valued functions on T.

Let $M^+(T)$ denote the space of positive Borel measures on T, and suppose that $\{Q_n(t)\}$ is a positive T-martingale. For $\sigma \in M^+(T)$ let $Q_n\sigma$ denote the random measure defined by $Q_n\sigma << \sigma$ and $\frac{dQ_n\sigma}{d\sigma}(t) := Q_n(t)$, $t \in T$. Denote the space of bounded continuous functions on T by $C_B(T)$. Then, essentially by the martingale convergence theorem, one obtains a random Borel measure $Q_\infty\sigma$ such that with probability one [K3]

$$\lim_{n\to\infty} \int_T f(t) Q_n\sigma\,(dt) = \int_T f(t) Q_\infty\sigma\,(dt), \qquad f \in C_B(T). \tag{1.1}$$

As suggested by the notation, one may view $\sigma \to \sigma_\infty \equiv Q_\infty\sigma$ as a random operator acting on $M^+(T)$. The following special case of multiplicative cascades is central to the general theory developed in [K3].

Independent cascades. Let $b \geq 2$ be a natural number, referred to as a *branching number*, and let $T = \{0, 1, \ldots, b-1\}^{\mathbf{N}}$ be the metric space for the ultrametric $\rho(s, t) = b^{-a(s,t)}$ where $a(s, t) = \inf\{n : s_n \neq t_n\}$, $s = (s_1, s_2, \ldots)$, $t = (t_1, t_2, \ldots)$. The countable set $T^* \equiv T^*(\infty) := \cup_{n=0}^{\infty} T^*(n)$, where $T^*(n) := \cup_{k=1}^{n}\{0, 1, \ldots, b-1\}^k$, provides a convenient labelling of the

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vertices of the infinite b-ary tree. It is sometimes convenient to adjoin a root vertex denoted $\varnothing$ to T^*. For $t \in T$, $n \geq 1$, write $t|n := (t_1, t_2, \dots, t_n) \in T^*$, $t|0 := \varnothing$. For $\gamma = (\gamma_1, \dots, \gamma_n) \in T^*$, $j \in \{0, 1, \dots, b-1\}$, define $|\gamma| := n$ and $\gamma * j := (t_1, \dots, t_n, j)$. Also denote the nth generation partition by $\Delta_\gamma := \{t \in T : t|_{|\gamma|} = \gamma\}$. In this context the *cascade generators* are furnished by a (denumerable) family of i.i.d. nonnegative mean-one random variables $\{W_\gamma : \gamma \in T^*\}$ indexed by the tree and defined on a probability space $(\Omega, \mathscr{F}, P)$. With this, let

$$P_n(t) := \sum_{|\gamma|=n} W_\gamma 1_{\Delta_\gamma}(t). \tag{1.2}$$

Now the *homogeneous independent cascade* is the positive T-martingale defined by $Q_n = P_1 P_2 \cdots P_n$ with respect to $\mathscr{F}_n := \sigma\{W_\gamma : |\gamma| \leq n\}$. Define $\lambda_\infty := Q_\infty \lambda$, where here and throughout λ denotes the Haar measure on T. It may happen that $\lambda_\infty = 0$ a.s., referred to as a *degeneracy*. The onset of nondegeneracy may be viewed as a critical phenomenon associated with the cascade. The fundamental theorem of [KP] on the structure of homogeneous independent cascades provides necessary and sufficient conditions on the distribution of the generators for (i) *nondegeneracy and,* (ii) *divergence of moments of* $\lambda_\infty[0, 1]$, and provides in all cases of nondegeneracy (iii) *the Hausdorff dimension of the support.* Replacement of the generators at each generation by i.i.d. nonnegative random vectors $\mathbf{W} := (W_0, \dots, W_{b-1})$, $E\frac{1}{b}\sum W_j = 1$, is possible without substantial changes in the conditions [N]. Moreover, this makes it possible to include the random distribution functions studied by [DF]. The results described in this note provide an entirely probabilistic approach to the computation of the fine-scale properties of random cascades, which also includes a simple new solution to problems (i)–(iii) for independent cascades. As will be seen, the power of this approach for more general cascades is that fine-scale computations are reduced to the law of large numbers (ergodic theory) problems.

Much of the recent focus of the study of independent cascades, both theoretically and empirically, has concerned various characteristics of their singularity structure, e.g. [HW, CK, F, PW, MS, O]. In this regard, for an arbitrary Borel measure σ, one has a unique disintegration $\sigma(\cdot) = \int \sigma_\alpha(\cdot)\nu(d\alpha)$ (properly interpreted), where $\int_{[0,\beta]} \sigma_\alpha \nu(d\alpha)$ is supported by a set of Hausdorff dimension no larger than β, and $\int_{(\beta,\infty]} \sigma_\alpha(B)\nu(d\alpha) > 0 \Rightarrow \dim(B) > \beta$ [C, KK]. The measure ν is called the *dimension spectrum.* Each spectral mode σ_α is supported on a set of dimension at most α. If ν has an atom at α, then σ_α is *unidimensional*; i.e. the dimension spectrum of σ_α is a Dirac point mass. It follows from [KP] that the (homogeneous) independent cascades are a.s. unidimensional. The richness of extensions of the cascade theory to dependent cascades is partly reflected in an often natural nontrivial spectral disintegration.

2. Main results

The original proof of the Kahane-Peryière theorem is based on a combination of probabilistic and analytic computations which make strong use of the *statistical independence.* The point of focus in [KP] is a *distributional fixed-point equation* for the total mass of the cascade. Extensions to a limited class of dependent cascades, namely, finite-state Markov generators, have been found along similar lines in [WW1]. However, even in the countably infinite-state Markov context, the fixed-point analysis does not readily extend to dependent generators. It is interesting to note that the fixed-point equation also arises in certain interacting particle context, [HL, DL]. The approach described in this note may be of independent interest when applied to the fixed-point problem under dependence.

To define the general *dependent random cascade,* we begin with a probability measure $p_0(dx)$ and a collection of mean-one transition probability kernels $q_n(dx|x_0, \ldots, x_{n-1})$, on $\mathscr{B}[0, \infty)$, $x_i \geq 0$, $n \geq 1$. Using the Kolmogorov extension theorem, one may construct a unique probability measure P on the product space $\Omega := [0, \infty)^{T^*(\infty)}$, $:= := \mathscr{B}^{T^*(\infty)}[0, \infty)$ together with the coordinate projection random variables $W_\gamma(\omega) := \omega_\gamma$ such that (i) $W_\varnothing$ has distribution $p_0(dx)$; (ii) for any $t \in T$, the conditional distribution of $W_{t|n+1}$ given W_γ, $|\gamma| \leq n$, is $q_{n+1}(dx|W_{t|0}, W_{t|1}, \ldots, W_{t|n})$; and (iii) for $s \in T$, $s|n+1 \neq t|n+1$, $W_{s|n+1}$ is conditionally independent of $W_{t|n+1}$ given W_γ, $|\gamma| \leq n$ [WW2]. We refer to this model as the *cascade generator* corresponding to the given transition kernels q_n and initial distribution p_0. By relabeling the states, the two-state "Markov chains on trees" constructed in [Pr, Sp] may be adapted as a special case to illustrate this general setting.

Given a cascade generator, one can define a positive T-martingale by applying the formula (1.2). The resulting random measure $\lambda_\infty := Q_\infty \lambda$ will be referred to as the *dependent cascade.* Our approach to the study of cascades in this generality is based on three elements, namely, (i) *a weight system perturbation,* (ii) *a size-bias transform*, and (iii) *a general percolation method.*

Definition 2.1. A *weight system* is a family $\mathbf{F}$ of real-valued functions $F_\gamma : \Omega \to [0, \infty)$ indexed by the tree, where for each $\gamma \in T^*$, F_γ is $\sigma\{W_{\gamma|j} : j \leq |\gamma|\}$ measurable, such that $Q_{\mathbf{F}, n}(t) := \sum_{|\gamma|=n} \prod_{j \leq n} (W_{\gamma|j}) F_\gamma 1_{\Delta_\gamma}$ is a positive T-martingale, referred to as a *weighted cascade.* A weight system $\mathbf{F}$ is called a *weight decomposition* in the case $\mathbf{F}^c := \{1 - F_\gamma : \gamma \in T^*\}$ is also a weight system.

Note that a weight system is a weight decomposition if and only if the weights are bounded between 0 and 1.

Assuming that one already has a nondegenerate limit cascade, Peyrière [P] defines a probability $\mathscr{Q}$ on $\Omega \times T$, named the *Peyrière probability* in [K1], for the joint distribution of a randomly selected path and the cascade generators along the path. This probability plays an important role in the analysis of the structure of independent cascades *provided* nondegeneracy has been established. The following probability is a useful extension of this notion in two directions: namely, (i) it does not require an a priori nondegeneracy condition; and (ii) it permits perturbations by a weight system. For a given weight system $\mathbf{F}$, let $Q_{\mathbf{F}, n}(t) := b^{-n} \prod_{j \leq n} (W_{t|j}) F_{t|n}$, $t \in T$, $n \in \mathbf{N}$. Define a sequence $\mathscr{Q}_{\mathbf{F}, n}$ of measures on $\Omega \times T$ by

$$(2.1) \qquad \int_{\Omega \times T} f(\omega, t) \mathscr{Q}_{\mathbf{F}, n}(d\omega \times dt) := E_P \int_T f(\omega, t) Q_{\mathbf{F}, n}(t) \lambda(dt),$$

for bounded measurable functions f. Then one normalizes the masses of the $\mathcal{Q}_{\mathbf{F},n}$ by a factor $Z_\varnothing := EW_\varnothing F_\varnothing$ and extends to a probability $\mathcal{Q}_{\mathbf{F}}$ using the Kolmogorov extension theorem. One requires $Z_\varnothing > 0$ here and throughout.

The third ingredient to this theory is a generalization of an idea considered without proof in [K1], which may be viewed as a percolation method; see [WW2] for proofs. By independently pruning the tree, one studies the critical parameters governing the survival of mass, i.e. nondegeneracy of the percolated cascade, to determine dimension estimates on the support of the cascade. This is similar to an idea explored in [L] but differs by the distinction between *locations* and *amounts* of positive mass.

The following results provide the foundations for the general theory being announced here.

Theorem 2.1 (A Lebesgue decomposition). *Let π_Ω denote the coordinate projection map of $\Omega \times T$ onto Ω. Then*

$$d\mathcal{Q}_{\mathbf{F}} \circ \pi_\Omega^{-1} = Z_\varnothing^{-1}\lambda_{\mathbf{F},\infty}(T)1(\lambda_{\mathbf{F},\infty}(T) < \infty)dP + 1(\lambda_{\mathbf{F},\infty}(T) = \infty)d\mathcal{Q}_{\mathbf{F}} \circ \pi_\Omega^{-1}$$

where $\lambda_{\mathbf{F},\infty} = Q_{\mathbf{F},\infty}\lambda$.

Theorem 2.2 (A size-biased disintegration). *Given a weighting system* $\mathbf{F}$,

$$\mathcal{Q}_{\mathbf{F}}(d\omega, dt) = P_{\mathbf{F},t}(d\omega)\lambda(dt),$$

where

$$\frac{dP_{\mathbf{F},t}}{dP}|_{\mathcal{F}_n} = \prod_{j\le n}(W_{t|j})F_{t|n}.$$

Theorem 2.3 (A submartingale bound). *Let $\mathcal{F}_{t,n} := \sigma\{W_{t|i}, W_\gamma : i \ge 0, |\gamma| \le n\}$. Fix $c_k \ge 0$, such that c_k is $\mathcal{F}_{t,0}$-measurable, $E_{\mathcal{Q}_{\mathbf{F}}}\sum_k c_k < \infty$. Given a weight system $\mathbf{F}$, letting $\lambda_{\mathbf{F},n} = Q_{\mathbf{F},n}\lambda$, one has for arbitrary $t \in T$*

$$b^{-n}\prod_{i\le n}(W_{t|i})F_{t|n} \le \lambda_{\mathbf{F},n}(T) \le b^{-n}\prod_{i\le n}(W_{t|i})F_{t|n} + \sup_j\Big(\frac{\prod_{i\le j}(W_{t|i}b^{-1})}{c_j}\Big)M_n,$$

where

$$M_n = \sum_{j=0}^{n-1} c_j b^{-(n-j)} \sum_{\substack{|\gamma|=n-j\\ \gamma(1)\ne t(j+1)}} \prod_{i=1}^{n-j}(W_{[t|j]*[\gamma|i]})F_\gamma$$

is a nonnegative $P_{\mathbf{F},t}$-submartingale with respect to $\mathcal{F}_{t,n}$, whose Doob decomposition has the predictable part $A_n = \frac{b-1}{b}\sum_{j\le n-1} c_j$.

Theorem 2.4 (A percolation method). *Let $Q_{\beta,n}(t)$, $t \in T$, be the β-model defined by the cascade with independent generators*

$$W_\gamma := B_\gamma = \begin{cases} b^\beta & \textit{w. prob. } b^{-\beta}, \\ 0 & \textit{else} \end{cases}$$

and independent of the weighted cascade $Q_{\mathbf{F},n}(t)$. *Then* $Q_n(t) := Q_{\beta,n}(t)Q_{\mathbf{F},n}(t)$ *is a weighted cascade, with weights* $\mathbf{F}_\beta$ *defined by* $(F_\beta)_\gamma := B_\gamma F_\gamma$, *such that for* $\sigma \in M^+(T)$

$$Q_\infty\sigma = Q_{\beta,\infty}(Q_{\mathbf{F},\infty}\sigma) \quad a.s.$$

One way in which the significance of this approach is illustrated is by the existence of the following natural weight systems.

Theorem 2.5 (Kahane decomposition). *The Kahane decomposition* [K2, K4], $Q_n(t) = Q_n'(t) + Q_n''(t)$, *is equivalent to a weight decomposition with respect to the weights defined by* $F_{t|n} = \frac{Q_n'(t)}{Q_n(t)}$, $t \in T$, *and* $1 - F_{t|n}$ *for* $Q_n'(t)$, $Q_n''(t)$ *respectively.*

Theorem 2.6 (Dimension decomposition). *The decomposition* [KK, C] $\lambda_\infty = \int (\lambda_\infty)_\alpha \nu\,(d\alpha)$ *is equivalent to a weight decomposition with respect to the weight systems* $\mathbf{F}_s := \{F_{s,\gamma} : \gamma \in T^*\}$, $s \in [0, 1]$, *defined by*

$$F_{s,\gamma} = \frac{E[\nu_\gamma([0, s])|\mathscr{F}_n]}{\lambda_n(\Delta_\gamma)},$$

where

$$\nu_\gamma([0, s]) := \int_{[0,s]} (\lambda_\infty)_\alpha(\Delta_\gamma)\nu\,(d\alpha), \qquad n = |\gamma|.$$

Remark 1. Two interesting classes of dependent cascades are: (i) *Markov cascades and,* (ii) *exchangeable cascades.* One may compute the weighting decompositions for Markov cascades, for example, in terms of *harmonic measures* corresponding to hitting probabilities of the *survival classes* introduced in [WW1]. Also for exchangeable cascades one may compute these weighting decompositions in terms of conditioned de Finetti measures applied to a partition of $M^+(T)$.

Remark 2. One may show that the maps $s \to F_{s,\gamma}$ are nondecreasing in s and the Lebesgue-Stieltjes measures associated with the maps $s \to \int F_{s,t|n}\lambda_n\,(dt)$ converge vaguely to the spectral measure ν. In fact, if the map $s \to F_{s,\gamma}$ is absolutely continuous with respect to some measure σ, then $t \to \int \frac{dF_{t,\gamma}}{d\sigma}\lambda_n\,(d\gamma)$ is itself a positive $T = [0, \infty)$-martingale. In this case the percolation method of Theorem 2.4 may be applied to compute the dimension spectrum. This, in fact, illustrates a more general notion of *differentiable weights* which are introduced to compute more detailed dimension spectra in [WW3].

Finally, the scope of the theory is well illustrated by the following approach to the Kahane-Peyrière theorem. In particular, the suitability of this method for more general cascades is made transparent by the suppressed role of independence beyond the martingale structure and/or the general ergodicity under the size-biased distribution.

Consider the independent cascade. Let W denote a generic generator with distribution $q(dx)$. We write $\mathbf{F} = 1$ to denote the case of unit weights.

A sufficient condition for nondegeneracy. Let us first show how it follows that $E_P W \log_b W < 1$ is a sufficient condition for nondegeneracy. Choose $0 < \log_b c < 1 - E_P W \log_b W$. Fix $t \in T$. Under $P_{1,t}$, for γ's along the t-path, i.e. $\gamma = t|j$, some j, the W_γ's are i.i.d. with distribution $xq(dx)$; thus the name "sized-biased". For γ's off of the t-path, i.e. $\gamma \neq t|j$, the W_γ's are i.i.d. with distribution $q(dx)$. It follows from the SLLN (i.e. ergodicity) that $P_{1,t}$-a.s., $\sqrt[j]{\frac{c^j}{b^j}\prod_{i=1}^{j} W_{t|i}} \to \frac{c}{b}e^{E_P W \log W}$. Now use Theorem 2.3, with $c_j = c^{-j}$, to conclude that $\lambda_\infty(T) < \infty$, $P_{1,t}$-a.s., and therefore, by Theorem 2.2, after integrating out t, $\lambda_\infty(T) < \infty$, $\mathscr{Q}_1$-a.s. Now apply Theorem 2.1 to see that $E\lambda_\infty(T) = 1$.

A necessary condition for nondegeneracy. To see why the entropy condition $E_P W \log_b W < 1$ is also necessary for nondegeneracy, suppose that $E_P W \log_b W \geq 1$, $P_t(W = b) < 1$. Fix $t \in T$. If $E_P W \log_b W > 1$, then one obtains $\lambda_n(T) \to \infty$ $P_{1,t}$-a.s. directly from Theorem 2.3. If $E_P W \log_b W = 1$ but $P_t(W = b) < 1$, then, since under the size-biasing $W > 0$ a.s., one has that $\limsup \log \lambda_n(T) = \infty$, $P_{1,t}$-a.s. by the Chung-Fuchs null-recurrence criterion for the random walk along the t-path. Integrate out t and use Theorems 2.2 and 2.1 (in that order) to conclude that $\lambda_\infty(T) = 0$, P-a.s. The special case of the degeneracy $P_t(W = b) = 1$, i.e. $P(W = b) = \frac{1}{b} = 1 - P(W = 0)$, may be handled similarly or directly from the theory of branching processes.

In a similar manner one may obtain the Kahane-Peyrière divergence of moments criterion from Theorems 2.3, 2.2, and 2.1 (in that order). The condition for nondegeneracy and the percolation method of Theorem 2.4 may then be combined to obtain the Hausdorff dimension of the support of λ_∞ in the manner first noted in [K1].

The detailed proofs for the underlying theory and applications to general classes of dependent cascades appear in the companion paper [WW3].

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