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ON THE GEOMETRIC AND TOPOLOGICAL RIGIDITY OF HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. A homotopy equivalence between a hyperbolic 3-manifold and a closed irreducible 3-manifold is homotopic to a homeomorphsim provided the hyperbolic manifold satisfies a purely geometric condition. There are no known examples of hyperbolic 3-manifolds which do not satisfy this condition.

One of the central problems of 3-manifold topology is to determine when a homotopy equivalence between two closed orientable irreducible 3-manifolds is homotopic to a homeomorphism. If one of these manifolds is S^3, then this is Poincaré's problem. The results of [Re], [Fr], [Ru], [Bo], and [HR] (see also [Ol]) completely solve this problem for maps between lens spaces. In particular there exist nonhomeomorphic but homotopy equivalent lens spaces (e.g. L(7,1) and L(7,2)), and there exist self-homotopy equivalences not homotopic to homeomorphisms (e.g. the self-homotopy equivalence of L(8,1) whose π_1-map is multiplication by 3). By Waldhausen [W] (resp. Scott [S]) a homotopy equivalence between a closed Haken 3-manifold (resp. a Seifert-fibred space with infinite π_1) and an irreducible 3-manifold can be homotoped to a homeomorphism. By Mostow [M] a homotopy equivalence between two closed hyperbolic 3-manifolds can be homotoped to a homeomorphism and in fact an isometry. However, the general case of homotopy equivalence between a hyperbolic 3-manifold and an irreducible 3-manifold remains to be investigated. These problems and results should be contrasted with the conjecture [T] that a closed irreducible orientable 3-manifold is either Haken, or Seifert fibred with infinite π_1, or the quotient of S^3 by an orthogonal action, or the quotient of $\mathbb{H}^3$ via a cocompact group of hyperbolic isometries.

Theorem 1 [G2]. *Let N be a closed, orientable, hyperbolic 3-manifold containing an embedded hyperbolic tube of radius $(\log 3)/2 = .549306\ldots$ about a closed geodesic. Then:*

(i) *If $f : M \to N$ is a homotopy equivalence where M is an irreducible 3-manifold, then f is homotopic to a homeomorphism.*
(ii) *If $f, g : M \to N$ are homotopic homeomorphisms, then f is isotopic to g.*
(iii) *The space of hyperbolic metrics on N is path connected.*

Remarks. (i) Thus N is both topologically and geometrically rigid provided it satisfies the purely geometric condition of having a modest-sized tube about a geodesic. Actually the conclusion of Theorem 1 holds provided N satisfies a more general geometric/topological property called the *insulator condition.*

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(ii) Jeff Weeks's tube radius/ortholength program [We] has found one hyperbolic 3-manifold N (of volume 1.0149...) which fails to have a $(\log 3)/2$ tube. Again by Weeks, conclusions (i)–(iii) above are applicable to N because it satisfies the insulator condition.*

(iii) An application of the hyperbolic law of cosines shows that if the shortest geodesic δ in N has length > 1.353, then tube radius $(\delta) > (\log 3)/2$.

(iv) If N has a geodesic δ of length < 0.0978, then Meyerhoff's tube radius formula [Me, §3] implies that tube radius $(\delta) > (\log 3)/2$. Recently Gehring and Martin [GM1, 2] improved this number to 0.19. Combined with the work of Jorgenson [Gr], this shows that for any $n > 0$ there exist only finitely many hyperbolic 3-manifolds of volume $< n$ which can fail to satisfy the hypothesis of Theorem 1.

(v) Farrell and Jones [FJ] showed that if $f : M \to N$ is a homotopy equivalence between closed manifolds and N is a hyperbolic manifold of dimension ≥ 5, then f is homotopic to a homeomorphism.

The theme of the proof of Theorem 1 is to abstract the ideas in the following example to the setting of homotopy hyperbolic 3-manifolds.

Example 2. Let δ be a simple closed geodesic in the hyperbolic 3-manifold N. δ lifts to a collection $\Delta = \{\delta_i\}$ of hyperbolic lines in $\mathbb{H}^3$. To each pair δ_i, δ_j there exists the *midplane* D_{ij}, i.e. the hyperbolic halfplane orthogonal to and cutting the middle of the *orthocurve* (i.e. the shortest line segment) between δ_i and δ_j. Each D_{ij} extends to a circle λ_{ij} on S^2_∞, which separates $\partial\delta_i$ from $\partial\delta_j$. Now fix i. Let H_{ij} be the closed $\mathbb{H}^3$-halfspace bounded by D_{ij} containing δ_i. $W_i = \cap H_{ij} = D^2 \times \mathbb{R}$ is the *Dirichlet tube domain* associated to the geodesic δ_i. $\mathring{W}_i$ projects to an open solid torus in N containing δ as its core. In fact, $W = W_i/\langle\delta_i\rangle$ is a solid torus with boundary a finite union of totally geodesic polygons.

Definition 3. Let $\mathscr{A} = \{\lambda_{ij}\}$. We call the pair $(\pi_1(N), \mathscr{A})$ the *Dirichlet insulator family* associated to δ. It is *noncoalesceable* if for no i does there exist $\lambda_{ij_1}, \lambda_{ij_2}, \lambda_{ij_3}$ whose union separates the points of $\partial\delta_i$. A hyperbolic 3-manifold satisfies the *insulator condition* if the Dirichlet insulator family associated to some geodesic is noncoalesceable.

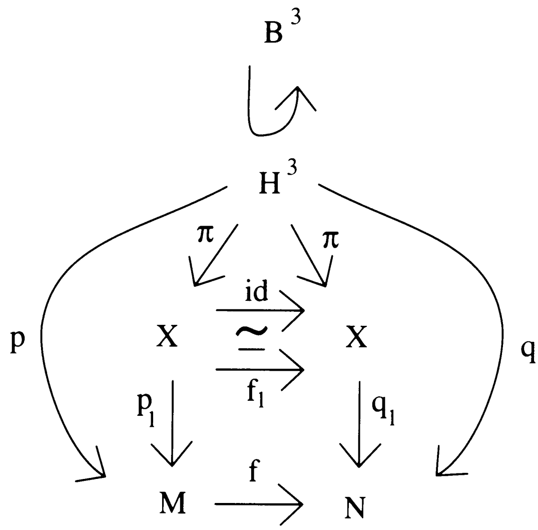
Conjecture 4. *The Dirichlet insulator family associated to a shortest geodesic in a closed orientable hyperbolic* 3*-manifold is noncoalesceable.*

Remarks. (i) The notion of Dirichlet insulator can be abstracted to the more general notion of *insulator* [G2].

(ii) If a geodesic δ has tube radius $> (\log 3)/2$, then its Dirichlet insulator family is noncoalesceable. The explanation boils down to the following observation in 2-dimensional hyperbolic geometry. If a geodesic γ_1 is at distance $\log(3)/2$ from a point x, then in the visual circle of x, γ_1 takes up exactly 120 degrees. Thus three geodesics of distance $> \log(3)/2$ from x cannot form a link around x.

*Note added in proof, June 21, 1994: Several weeks ago, Nathaniel Thurston, of the Geometry Center at the University of Minnesota, discovered what appear to be five additional hyperbolic 3-manifolds with a shortest geodesic which does not have a $\log(3)/2$ tube. There is strong evidence that these manifolds satisfy the insulator condition. Thurston made use of Robert Riley's POINCARÉ program as well as some ideas of Robert Meyerhoff and the author.

Proposition 5 [G1, 2]. *If $f : M \to N$ is a homotopy equivalence between the closed hyperbolic 3-manifold N and the irreducible 3-manifold M, then M and N are covered by the same closed hyperbolic manifold X. The covering map $p_1 : X \to M$ can be chosen so that the homotopy equivalence lifts and extends to a mapping $f_1 : X \to X$ homotopic to id_X. The group of covering transformations on $\mathbb{H}^3$ defined by $\pi_1(M)$ and $\pi_1(N)$ induce identical group actions on S^2_∞.*



The following proposition gives a criterion for showing that a homotopy equivalence can be deformed into a homeomorphism.

Proposition 6. *Let $f : M \to N$ be a homotopy equivalence between the closed orientable hyperbolic 3-manifold N and the irreducible 3-manifold M. If there exists a simple closed curve $\gamma \subset M$, a geodesic $\delta \subset N$ and a homeomorphism $k : (\mathbb{B}^3, p^{-1}(\gamma)) \to (\mathbb{B}^3, q^{-1}(\delta))$ such that $k \mid \partial\mathbb{B}^3 = \mathrm{id}$, then f is homotopic to a homeomorphism.*

Remarks. (i) Implicit in the statement of Proposition 6 is the definition of p, q and the identification of S^2_∞'s given in Proposition 5.

(ii) Said another way, f is homotopic to a homeomorphism provided the $\mathbb{B}^3$-link Δ is equivalent to the $\mathbb{B}^3$-link Γ. Here Δ is the preimage of δ in $\mathbb{H}^3$ extended to $\mathbb{B}^3$, and Γ is defined similarly. That these links are equivalent means that there exists a homeomorphism $k : (\mathbb{B}^3, \Gamma) \to (\mathbb{B}^3, \Delta)$ so that $k \mid S^2_\infty = \mathrm{id}$.

Theorem 7. *Let $f : M \to N$ be a homotopy equivalence, where M is a closed, connected, orientable, irreducible 3-manifold and N is a hyperbolic 3-manifold. If N possesses a geodesic δ with a noncoalesceable insulator family, then f is homotopic to a homeomorphism.*

Outline of the proof. To each smooth simple closed curve λ_{ij} in S^2_∞, there exists a lamination σ_{ij} by least area (with respect to the metric induced by M) planes in $\mathbb{H}^3$, with limit set λ_{ij} such that σ_{ij} lies in a fixed width hyperbolic regular neighborhood of the hyperbolic convex hull of λ_{ij}. Here $\{\lambda_{ij}\}$ is the $(\pi_1(N), \{\partial\delta_i\})$ and hence $(\pi_1(M), \{\partial\delta_i\})$ noncoalesceable insulator family.

Fix i. Let H_{ij} be the $\mathbb{H}^3$-complementary region of σ_{ij} containing the ends of δ_i. We show that $\cap_j H_{ij}$ contains a component $V_i = \mathring{D}^2 \times \mathbb{R}$ which projects to an open solid torus in M. Define γ to be the core of this solid torus and γ_i the lift which lives in V_i. The isotopy class of γ is independent of all choices, i.e. the metric on M and the choice of $\{\sigma_{ij}\}$ for a fixed metric. Let τ_0 be the link in X which is the preimage of γ, so $\{\gamma_i\}$ is also the set of lifts of components of τ_0 to $\mathbb{H}^3$. The Riemannian metric μ_0 on X induced from M and the hyperbolic metric μ_1 on X are connected by a smooth path μ_t of metrics. These metrics lift to $\pi_1(X)$ equivariant metrics $\tilde{\mu}_t$ on $\mathbb{H}^3$, so the above construction applied to the $(\pi_1(X), \{\partial\delta_i\})$ insulator family $\{\lambda_{ij}\}$ with respect to the $\tilde{\mu}_t$ metric yields a link τ_t in X. Since the isotopy class of τ_t is independent of t, τ_0 is isotopic to τ_1, the preimage of δ in X. We conclude that the $\mathbb{B}^3$-link Γ is equivalent to the $\mathbb{B}^3$-link Δ , and so by Proposition 6 f is homotopic to a homeomorphism.

Theorem 8. *If N is a closed, oriented, hyperbolic* 3*-manifold possessing a geodesic δ with a noncoalesceable insulator family and $f : N \to N$ is a homeomorphism homotopic to* id, *then f is isotopic to* id.

Idea of the proof. Let ρ_0 denote the hyperbolic metric on N. Let ρ_1 be the pull-back hyperbolic metric on N induced via f, which we can assume is a diffeomorphism. These metrics are connected by a family ρ_t. As in the proof of Theorem 7, to each ρ_t there is associated a simple closed curve γ_t where $\gamma_0 = \delta$ and $\gamma_1 = f^{-1}(\delta)$ and all of these γ_t's are isotopic. Therefore, f is isotopic to a map which fixes δ pointwise. A theorem of Siebenmann [BS] implies that f is isotopic to id.

Corollary 9. *If N satisfies the insulator condition, then*

$$\mathrm{Homeo}(N)/\,\mathrm{Homeo}_0(N) = \mathrm{Out}(\pi_1(N)) = \mathrm{Isom}(N).$$

Proof. Since a hyperbolic 3-manifold is a K(π ,1), homotopy classes of homeomorphisms are parametrized by Out($\pi_1(N)$). Mostow implies that each homotopy class is representable by a unique isometry. Theorem 8 implies that homotopy classes of homeomorphisms are the same as isotopy classes of homeomorphisms. □

Remark 10 (Why a coalesceable insulator is bad). It is possible that the $\cap_j H_{ij}$ resulting from the construction applied to a coalesceable insulator family would contain no $\mathring{D}^2 \times \mathbb{R}$ component. In fact, using the wrong metric, some hyperbolic plane P transverse to δ_i may be disjoint from $\cap_j H_{ij}$. This is the usual problem of a triangular prism formed by three minimal surfaces being obliterated upon change of metric.

Remark 11 (What Mostow does not say). If ρ_0 is a hyperbolic metric on N, then a nontrivial element α of $\pi_1(N)$ determines a geodesic δ_0 on N. Mostow's rigidity theorem does not rule out the possibility that with respect to a different hyperbolic metric ρ_1, the geodesic δ_1 associated to α would lie in a different isotopy class than δ_0. What Mostow does assert is that there exists a diffeomorphism $f : N \to N$, *homotopic* to id, such that $f(\delta_0) = \delta_1$. Theorems 7 and 8 show that if N satisfies the insulator condition, then δ_0 is isotopic to δ_1 and, further, that the diffeomorphism is isotopic to id. Said another way, Mostow asserts that hyperbolic structures are unique up to homotopy, while Theorem 1 asserts that under mild hypothesis a hyperbolic structure is unique up to isotopy.

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Bibliography

[Bo] F. Bonahon, *Diffeotopes des espaces lenticulaires*, Topology **22** (1983), 305–314.

[BS] F. Bonahon and L. Siebenmann (to appear).

[FJ] F. T. Farrell and L. Jones, *A topological analogue of Mostow's rigidity theorem*, J. Amer. Math. Soc. **2** (1989), 257–370.

[Fr] W. Franz, *Abbildungsklassen und fixpunktklassen dreidimensionalen linsenraume*, J. Reine. Angew. Math. **185** (1943), 65–77.

[G1] Gabai, *Homotopy hyperbolic* 3*-manifolds are virtually hyperbolic*, J. Amer. Math. Soc. **7** (1994), 193–198.

[G2] ———, *On the geometric and topological rigidity of hyperbolic* 3*-manifolds*, preprint.

[GM1] F. Gehring and G. Martin, *Commutators, collars and the geometry of Mobius groups*, J. d'Analyse (to appear).

[GM2] ———, *Torsion and volume in hyperbolic* 3*-folds*, in preparation.

[Gr] M. Gromov, *Hyperbolic manifolds according to Thurston and Jorgensen*, Sem. Bourbaki **32** (1979), 40–52.

[Me] R. Meyerhoff, *A lower bound for the volume of hyperbolic* 3*-manifolds*, Canada. J. Math. **39** (1987), 1038–1056.

[Mo] G. D. Mostow, *Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104.

[Ol] P. Olum, *Mappings of manifolds and the notion of degree*, Ann. of Math. (2) **58** (1953), 458–480.

[Re] K. Reidemeister, *Homotopieringe und Linsenraume*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 102–109.

[Ru] M. Rueff, *Beitrage zur untersuchung der abbildungen von mannigfaltigkeiten*, Compositio Math. **6** (1938), 161–202.

[S] P. Scott, *There are no fake Seifert fibred spaces with infinite* π_1, Ann. of Math. (2) **117** (1983), 35–70.

[T] W. P. Thurston, *Three-dimensional manifolds, Kleinian groups, and hyperbolic geometry*, Proc. Sympos. Pure Math., vol. 39, Amer. Math. Soc., Providence, RI, 1983, pp. 87–111.

[W] F. Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

[We] J. Weeks, *SnapPea: A computer program for creating and studying hyperbolic* 3*-manifolds*, available by anonymous ftp from `geom.umn.edu`.

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