it admits a definite and finite classification procedure... " (p. 120). Without denigrating the game of chess, it is foolish to compare the deep field of knot theory, including both the recent (meaning in the last fifteen years) important work in the field (cf. Thurston, Jones, Gordon-Luecke, Gabai) and its major unsolved problems (i.e., property P) to chess. The implication is that if we simply had more powerful computers, capable of implementing this unwielding algorithm, we could abandon the field.

In short, the book is a reasonably good explanation of a particular—and important—theorem in knot theory. It gives virtually no indication of any developments in knot theory since the solution of the classification problem (1979). With the exception of the section on normal surface theory, experts in the subject would do as well to read an expository article on Haken 3-manifolds by Waldhausen [W] and Hemion's original article [He], which appears as an appendix in the book. Nonexperts should refer to books that give a better idea of the scope of the field [A, R, BZ].

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Cohomological methods in transformation groups, by Christopher Allday and Volker Puppe. Cambridge Studies in Advanced Mathematics, vol. 32, Cambridge University Press, London, 1993, xi+470 pp., $69.95. ISBN 0-521-35022-0

One of the most fundamental structures in mathematics is that of symmetry. Understanding the role played by compact Lie groups as transformation groups is a central problem in mathematics.

In this book the authors describe a comprehensive *cohomological* approach to this question, based on the foundational work of P. A. Smith and A. Borel and its later development by G. Bredon, W. Browder, W.-Y. Hsiang, and D. Quillen, among others. They specialize in most situations to the case when the group in question is either a torus or an elementary abelian p-group, since it is well known that the methods are particularly successful for these groups.

Background

The starting point to the cohomological study of group actions was the method developed by P. A. Smith around 1940 (known today as "Smith Theory") whereby algebraic methods can be used to obtain important restrictions on group actions. For example, Smith proved (see [Bre])

Theorem 1. *If a finite group G acts freely on a sphere, then all its abelian subgroups are cyclic.*

Theorem 2. *If a finite p-group acts on a space X homeomorphic to the closed disc, then the fixed-point set has the* mod p *homology of a point and in particular is nonempty.*

These results clearly indicate that algebraic properties of a group are closely linked to its topological actions. This was confirmed later by examples of groups of composite order acting on a disc without stationary points (see [Bre] and the remarkable paper by R. Oliver [O]).

It was Armand Borel, however, who introduced the modern theory of fibrations and their associated spectral sequences in transformation groups [B]. This provided the technical framework needed for the proof of many important results. We briefly explain the key ideas, as they are still the core method in this area of mathematics.

If G is a topological group, let EG denote a free, contractible G-space. Now if G acts on a space X, then it will act diagonally on the product $EG \times X$, and this action will be free. The quotient space is denoted by $EG \times_G X$ and referred to as the *Borel Construction* on X. The projection onto the first factor induces a fibration

$$\begin{array}{ccc} X & \hookrightarrow & EG \times_G X \\ & & \downarrow \\ & & BG. \end{array}$$

The base of this fibration is known as the *classifying space* of G; if the group is discrete, then the cohomology of this space coincides with the usual group cohomology of G.

The first point to note is that the cohomology of $EG \times_G X$ can be approximated via the Serre spectral sequence using the cohomology of the base and the fiber (with possible twisting). The second point to note is that if F denotes the fixed-point set of the action, the inclusion induces a map $BG \times F \to EG \times_G X$ which under appropriate hypotheses can be understood. For example, if X is a compact n-manifold and $G = \mathbb{Z}/p$, then the map induced in mod p cohomology

$$H^*(EG \times_G X, \mathbb{Z}/p) \to H^*(BG \times F, \mathbb{Z}/p)$$

is an isomorphism above dimension n. Therefore, the spectral sequence affords direct information on the fixed-point set (see [Bre] for applications of this).

This was substantially improved on by Borel and Quillen [Q] in the form of the celebrated *Localization Theorem.* Namely, if G is an elementary abelian p-group and we consider $H^*(EG \times_G X, \mathbb{Z}/p)$ and $H^*(BG \times F, \mathbb{Z}/p)$ as modules over $\mathscr{R} = H^*(BG, \mathbb{Z}/p)$, then there is a multiplicative system $\mathscr{S} \subset \mathscr{R}$ such that the *localized map*

$$\mathscr{S}^{-1}H^*(EG \times_G X, \mathbb{Z}/p) \to \mathscr{S}^{-1}H^*(BG \times F, \mathbb{Z}/p)$$

is an isomorphism. There is an analogous form of this theorem for tori using rational coefficients. This powerful theorem (and its generalizations) provides important information concerning the fixed-point sets of actions of these groups on spheres, projective spaces, etc. (related to this, there are notable contributions by T. Chang, W.-Y. Hsiang, and T. Skjelbred; see [H]).

In addition, Quillen proved that for any compact Lie group G, the ring $H^*(EG \times_G X, \mathbb{Z}/p)$ has associated Krull Dimension equal to the rank of the largest elementary abelian p-subgroup of G fixing a point in X. Moreover, up to nilpotence its structure can be understood by using fixed-point data and restrictions to elementary abelian p-subgroups. This result has led to significant developments in group cohomology and modular representation theory (see [Ca]).

In the case $G = (\mathbb{S}^1)^k$, methods from rational homotopy theory have been particularly useful. Applying Sullivan's theory of minimal models allows for effective use of rational homotopy in an equivariant setting. This has made torus actions an especially fertile area of mathematics, where the authors and S. Halperin (as well as several others) have made valuable contributions.

Another successful point of view has been the use of equivariant Tate cohomology, first introduced by Swan [S]. The point is to obtain a modified model for equivariant cohomology which vanishes when the action is free. Browder [Bro] combined these techniques with exponents and the notion of degree to obtain substantial cohomological restrictions on finite group actions. Building on this, there has been an increasing use of methods from representation theory in group actions. Notions of complexity have had an important impact on certain problems; in particular, cohomological varieties and shifted subgroups have been quite useful (see [A]).

In summary, cohomological methods are an important and fundamental tool in transformation groups, with the necessary technical structure to incorporate diverse ideas and results from both algebra and topology.

The book under review is written in a lucid and careful style. All the topics previously mentioned are discussed (as well as many more), paying special attention to the key elements involved in the proofs. Alternate approaches are often discussed, and many interesting examples are provided. The authors have done an admirable job of explaining this area of mathematics. Thoughtful remarks are included in several places, there are exercises at the end of each chapter, and the references are abundant. Moreover, there are two appendices which provide much of the necessary background in commutative and differential algebra. The book is a natural continuation to Bredon's invaluable text [Bre], and it should prove useful to a broad spectrum of mathematicians.

Of course, the book has minor flaws, but they derive from the very nature of the subject. It is an area of mathematics that is difficult to penetrate for a beginner: presented as a closed field, it may seem arcane to many younger mathematicians. It would perhaps have been advisable to describe the material in the framework of interactions between algebra and topology, which is currently quite an active one (pertinent in some cases to problems in areas such as mathematical physics). Important recent work in homotopy theory has also been influenced by this area of mathematics (see [DW]). One should keep in mind that group actions are at the core of many of the ideas and methods needed to work in such an area of interphase, and this book records an important aspect of the subject.

A more concrete complaint may be lodged against the awkward piling on of hypotheses in some of the theorems. One often likes to be able to make sense of theorems in a book without constantly having to look back at the definitions. Faced with statements like "let X be a finite-dimensional CW-complex with FMCOT" or "$H_G^*(X, k)$ is a free R module if and only if X is TNHZ in $X_G \to BG$", the ordinary reader might recoil in horror. My advice is not to be frightened by this; these are in fact very natural hypotheses, but it is a task for a far better mathematical stylist than either this reviewer or the authors to find an aesthetic solution to this problem of presentation.

To conclude, this reviewer recommends the book by Allday and Puppe for anyone interested in compact transformation groups.

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