ory of decomposing algebras. A linear fractional parameterization of extensions results.

Some questions are left unanswered by this elegant construction. If a linear fractional transformation is present, then its defining matrix is related to a linear system. If the theory of unitary linear systems is to be taken seriously, then the construction needs to be supplied with scalar products which make the linear system unitary. When a unitary linear system has been found, the linear fractional transformation needs to be seen as a factorization in the theory of unitary linear systems.

Instructive applications of the present factorization theory are given to the Carathéodory-Toeplitz extension problem, the Nehari extension problem, and the Nevanlinna-Pick interpolation problem. The additional structure of a unitary linear system is present in all these examples. They suggest that the present band method can be restructured as a construction of unitary linear systems.

The authors are to be congratulated for an instructive formulation of the current status of a field which has a major impact on contemporary science and technology. Although the theories are not in final form, the methods applied are permanent because they are algorithms of computation. Further research can only deepen the understanding of why these methods are successful and widen the scope of their applications.

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Nilpotence and periodicity in stable homotopy theory, by Douglas C. Ravenel. Princeton University Press, Princeton, NJ, 1992, xiv + 209 pp., $24.95. ISBN 0-691-02572-X

The subject of homotopy theory, especially stable homotopy theory, was for many years guided by J. Frank Adams. In the final article in his selected works [1] he wrote: "At one time it seemed as if homotopy theory was utterly without system; now it is almost proved that systematic effects predominate." Adams was commenting on the influence of the results discussed in Ravenel's book, which are the subject of this review. The most striking of these results are due to Ethan Devinatz, Mike Hopkins, and Jeff Smith [2, 5] and were conjectured by Doug Ravenel [7] in the late seventies and early eighties.

To set the stage, recall that two continuous maps f and g from a space X to a space Y are *homotopic* if there is a continuous map $H : X \times [0, 1] \to Y$ agreeing with f on $X \times \{0\}$ and with g on $X \times \{1\}$. One often restricts attention to CW-complexes, i.e. spaces built in a systematic way by attaching cells. In stable homotopy theory, one is permitted to suspend a map $f : X \to Y$ as often as desired; its suspension $\Sigma f : \Sigma X \to \Sigma Y$ is defined in a natural way on the suspension of X, the "double cone" obtained from $X \times [0, 1]$ by collapsing the two copies $X \times \{0\}$ and $X \times \{1\}$ to separate points. Maps $f, g : X \to Y$ are called *stably homotopic* if, for some t, the iterated suspensions $\Sigma^t f$ and $\Sigma^t g$ are homotopic.

To make matters more precise, consider a *self-map* of a finite CW-complex X, by which we shall mean a map

$$\Sigma^d X \xrightarrow{f} X$$

for some $d \geq 0$. Especially, if X is the n-sphere S^n, then (since the suspension of S^n is S^{n+1}) f is a map $S^{n+d} \xrightarrow{f} S^n$. Now one can iterate self-maps up to suspension by forming composites

$$\cdots \to \Sigma^{2d} X \xrightarrow{\Sigma^d f} \Sigma^d X \xrightarrow{f} X$$

to obtain further self-maps which we denote by f^2, f^3, ... Call a self-map f *nilpotent* if some iterate $f^t : \Sigma^{td} X \to X$ is stably homotopic to a constant map; if this is not the case, call f *periodic*. In case $X = S^n$, one has a twenty year-old theorem due to Goro Nishida [6]:

Theorem. *Each map* $f : S^{n+d} \to S^n$ *with* $d > 0$ *is nilpotent.*

How can one hope to generalize Nishida's theorem to determine when a self-map $f : \Sigma^d X \to X$ is nilpotent for a general finite CW-complex X? The idea is a basic one in algebraic topology. One should search for a suitable *homology theory* $E_*(\cdot)$, meaning a sequence of functors $E_i(\cdot)$ from spaces to abelian groups satisfying the usual Eilenberg-Steenrod axioms for homology theory, except for the dimension axiom (so one permits $E_i(\text{point})$ to be nonzero even if $i \neq 0$). As for ordinary homology, one then defines reduced homology groups of a nonempty space by putting

$$\overline{E}_i(X) = \ker\big(E_i(X) \to E_i(\text{point})\big)$$

for the unique map from X to a one-point space. Since there is a suspension isomorphism

$$\overline{E}_i(\Sigma^d X) \cong \overline{E}_{i-d}(X),$$

the homomorphisms $E_*(f) : \overline{E}_i(\Sigma^d X) \to \overline{E}_i(X)$ induced by a self-map $f : \Sigma^d X \to X$ can be viewed as an endomorphism of the graded abelian group $\overline{E}_*(X)$. Now if f is nilpotent, as defined above, then (by the homotopy axiom for a homology theory) so is the endomorphism $E_*(f)$; we seek a homology theory which is reasonably easy to compute and for which the converse is true. The remarkable fact is that such a homology theory exists.

Nilpotence Theorem. *There is a homology theory* $\mathrm{MU}_*(\cdot)$, *known as the complex bordism theory, for which a self-map* $f : \Sigma^d X \to X$ *of a finite CW-complex is nilpotent if and only if the endomorphism* $\mathrm{MU}_*(f)$ *of* $\overline{\mathrm{MU}}_*(X)$ *is nilpotent.*

This theorem requires a delicate and lengthy argument, given by Devinatz, Hopkins, and Smith in [2]. Nishida's nilpotence theorem for self-maps $\Sigma^d S^n \xrightarrow{f} S^n$ with $d > 0$ is an immediate corollary, since $\mathrm{MU}_*(f)$ is then the zero homomorphism. In fact, two further forms of the nilpotence theorem are proved in [2], being natural generalizations of Nishida's theorem in two different directions. These "smash product" and "ring spectrum" forms of the nilpotence theorem are equivalent and imply the "self-map" form of the theorem discussed above. The subject is rounded out by the periodicity theorem of Hopkins and Smith [5] mentioned below and is elegantly summarized by Hopkins in [3].

The nilpotence theorem suggests that the complex bordism theory mirrors stable homotopy theory with considerable accuracy. For a finite CW-complex X, $\mathrm{MU}_*(X)$ is a graded module over the ring $\mathrm{MU}_*(\text{point}) = \mathrm{MU}_*$. Indeed, from cobordism theory one has long known that MU_* is a polynomial ring,

$$\mathrm{MU}_* \cong \mathbb{Z}[x_2, x_4, \dots],$$

the cobordism ring of smooth compact manifolds for which the tangent bundle (perhaps after adding a trivial bundle) has a complex structure. Each polynomial generator x_{2n} is the cobordism class of a suitable manifold M^{2n} with a complex structure on its stable tangent bundle (these manifolds can be taken to be finite disjoint unions of smooth complex projective varieties).

It is tempting to elaborate on the further structure of complex bordism theory, but I hope it will suffice to say that $\mathrm{MU}_*(X)$ is often computable, that there is a well-developed theory of cohomology operations which leads to a version of the Adams spectral sequence for computing stable homotopy groups, and that there is a close connection with the subject of one-parameter formal group laws. Indeed, this connection has been crucial for many advances over the last twenty-five years, and it is the source for the recent work by Gross and Hopkins [4] on Lubin-Tate moduli spaces and their applications to stable homotopy theory.

To state the periodicity theorem, we must point out that along with complex bordism $\mathrm{MU}_*(\cdot)$ one frequently studies a great many associated homology theories. When one's interest is focused on a fixed prime p, a sharper view of stable homotopy theory is provided by a sequence of homology theories $\mathrm{K}(n)_*(\cdot)$ for $n \geq 0$, called Morava K-theories. The first of these is just rational homology, $\mathrm{K}(0)_*(X) = H_*(X; \mathbb{Q})$, while for $n > 0$ the homology of a point is given by $\mathrm{K}(n)_*(\text{point}) = \mathbb{F}_p[v_n, v_n^{-1}]$ with v_n of degree $2(p^n - 1)$ (so these are periodic homology theories). Now let X be a finite CW-complex; we say that X has *type n* at p if $\overline{\mathrm{K}(n)}_*(X) \neq 0$, while for $m < n$ we have $\overline{\mathrm{K}(m)}_*(X) = 0$. Each finite CW-complex X with $\overline{H}_*(X; \mathbb{F}_p) \neq 0$ has type n at p for some n ($0 \leq n < \infty$); there do exist finite CW-complexes of type n for all n.

Periodicity Theorem. *Let X be a finite CW-complex of type n at the prime p. Then there is a self-map $f : \Sigma^{d+i}X \to \Sigma^i X$ for some $i \geq 0$ such that $\mathrm{K}(n)_*(f)$ is an isomorphism and $\mathrm{K}(m)_*(f) = 0$ for $m > n$. Moreover, the self-map f is essentially unique* (*after sufficient iteration and suspension*), *and for $n > 0$ the self-map can be chosen so that $\mathrm{K}(n)_*(f)$ is multiplication by a power of v_n.*

It is remarkable that there are finite CW-complexes of type n at p for all n, and all the more so that they admit self-maps detected by the Morava K-theories $\mathrm{K}(n)_*(\cdot)$; reasonably easy arguments suffice only for small values of n. The periodicity theorem could have been stated in terms of $\mathrm{MU}_*(X)$, but this is seldom done since the Morava K-theories are far more efficient for this purpose (even though they still lack a geometric definition for $n > 1$).

Ravenel's book presents these results with considerable enthusiasm and with a style and organization that readers will surely find illuminating. The first chapter provides an introduction to the subject, while background on homotopy theory and complex bordism can be located in the appendices as well as in the author's previous book [8]. A further appendix covers results due to Jeff Smith on the representations of the symmetric group which are needed for the proof of the periodicity theorem. The rest of the book patiently develops the subject, culminating in the proofs of the three forms of the nilpotence theorem and the periodicity theorem as well as a number of further results. The author has done a marvelous job of making difficult material accessible and inviting.

The book does contain a number of misprints and other slips. The proofs of Theorem 3.4.2 on pages 35–36 and of Corollary 5.1.5 on pages 50–51 need some reworking. Page entries in the index need to be reduced by two. Readers may wish to obtain a helpful errata listing by contacting the author at `drav@troi.cc.rochester.edu`.

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Mathematical elasticity. Vol. 1. Three-dimensional elasticity, by Phillippe Ciarlet. Studies in Mathematics and Its Applications, vol. 20, Elsevier Science Publishers, Amsterdam, 1988, 451 pp., \$107.25. ISBN 0-444-70259-8

The mathematical foundations of elasticity theory were established in large part during the nineteenth century by mathematicians such as Euler, Cauchy,