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Clifford algebras and Dirac operators in harmonic analysis, by John E. Gilbert and Margaret A. M. Murray. Cambridge Studies in Advanced Mathematics, vol. 26, Cambridge University Press, Cambridge, 1991, vi+334 pp., $75.00. ISBN 0-521-34654-1

Two generalizations come to mind for the familiar constructions of complex numbers as 2×2 matrices of reals and quaternions as 2×2 matrices of complex numbers. On the one hand, there is the construction that leads to the general class of *composition algebras* and its elegant but limited theory, i.e., normed algebras in which the norm of a product is the product of the norms, and on the other, the various constructions yielding the much more useful Clifford algebras.

The purpose of this book is to provide a systematic and, for the most part, self-contained introduction to the use of Clifford algebras and Dirac operators in analysis. These applications are often deep and technically complicated, including, for example, the Cauchy integral and Hilbert transform theory on multidimensional Lipschitz domains that was originated by Calderon [1] and continued in [2], [3], and [4], the realizations of discrete series for semisimple Lie groups [5], and the Atiyah-Singer index theorem [6]. One of the authors' goals was to make such material more accessible to classically trained analysts. To this end, they have given a very thorough treatment of the underlying algebraic aspects of the theory, including numerous examples and the main structural results on Clifford algebras and the various associated spin groups. On the analytic side, which is more tersely presented, they have concentrated on the applications to singular integral theory, generalizations of Hardy space theory, problems in representation theory involving harmonic functions, and generalized Cauchy Riemann systems associated with the Dirac operator.

In the book under review, A *Clifford algebra* is an associative algbera $\mathscr{A}$ with identity that is specified by a finite-dimensional real or complex vector space V, a symmetric bilinear form B on V, and a linear embedding ν of V into $\mathscr{A}$ such that $\mathscr{A}$ is generated by $\nu(V)$ and

$$\nu(x)^2 = -B(x, x) \tag{1}$$

for all x in V, scalars being identified with multiples of the identity in $\mathscr{A}$. Familiar polarization identities show that (1) is equivalent to the statement

$$\nu(x)\nu(y) + \nu(y)\nu(x) = -B(x, y) \tag{2}$$

for all $x, y \in V$. This definition, apart from the unnecessary but sensible restriction on the underlying fields, is more general than the one found in some familiar treatments of the subject [7], [8], [9]. Here there may be nonisomorphic Clifford algebras associated with a given pair V, B; however, there is a universal Clifford algebra for V, B that is unique up to isomorphism. The Clifford algebra $\mathscr{A}$, ν is *universal* for V, B if for every Clifford algebra $\mathscr{A}'$, ν' for V, B there is an algebra homomorphism $\mu : \mathscr{A} \to \mathscr{A}'$ such that $\mu\nu = \nu'$; this is equivalent to the statement that $\mathscr{A}$ is freely generated except for the relations given by (2). If n is the dimension of V, it is easy to see that the dimension of any Clifford algebra $\mathscr{A}$ for V, B is at most 2^n and that $\mathscr{A}$ is a universal Clifford algebra iff its dimension is 2^n.

For example, the standard quaternions i, j, k span a real vector space of dimension 3 and with ν the identity satisfy (1) relative to the usual quaternionic norm; however, there is an additional relation, namely, $ij = k$, and the dimension of the quaternion algebra is 4 rather than 8. But this algebra is universal for the subspace spanned by i, j.

Assume that $\mathscr{A}, \nu$ is a universal Clifford algebra for a real vector space V with a nondegenerate form B. Let Ω be an open set in V and $e_1, \dots, e_n$ an orthogonal base for V. Then the *Dirac operator* associated with this data is the first-order differential operator on $C^\infty(\Omega, \mathscr{A})$ defined by

$$D = \sum_{j=1}^{n} B(e_j, e_j) e_j \partial_j. \tag{3}$$

It is independent of the coordinate system, and its square is minus the Laplacian associated with the same data.

The authors give three proofs for the existence of universal Clifford algebras. The first construction realizes the algebra as an iterated tensor product of two-dimensional algebras, the second is a realization by linear operators on the exterior algebra associated with the underlying vector space, and the third is a familiar construction yielding the desired algebra as a quotient of the tensor algebra over the underlying space.

The following remarks provide a basis for an alternate inductive construction for universal Clifford algebras. Suppose that $\mathscr{A}$ is a universal Clifford algebra for the pair V, B with embedding $\nu : V \to \mathscr{A}$. Then there is a canonical automorphism $a \to a'$ of $\mathscr{A}$ of period 2, called the *principal* automorphism, such that $\nu(x)' = -\nu(x)$ for all $x \in V$. Given this, simple calculations show that

$$\mathscr{A}^+ = \left\{ \begin{pmatrix} a & b \\ -b' & a' \end{pmatrix} : a, b \in \mathscr{A} \right\} \tag{4}$$

is a matrix algebra that contains a copy of $\mathscr{A}$ on the diagonal. Its dimension is twice that of $\mathscr{A}$, and again simple computations show that the formula

$$\begin{pmatrix} a & b \\ -b' & a' \end{pmatrix}' = \begin{pmatrix} a' & -b' \\ b & a \end{pmatrix}$$

extends the principal automorphism of $\mathscr{A}$ to an automorphism of $\mathscr{A}^+$ that also has period 2. If V^+ is spanned by V and a vector $e \notin V$, then there is a symmetric bilinear form B^+ on V^+ such that e is orthogonal to V and $B(e, e) = 1$. It follows that the equation

$$\nu^+(x + \lambda e) = \begin{pmatrix} \nu(x) & \lambda \\ -\lambda & -\nu(x) \end{pmatrix} \tag{5}$$

defines a linear map $\nu^+ : V^+ \to \mathscr{A}^+$ that satisfies (1). Therefore, $\mathscr{A}^+, \nu^+$ is a universal Clifford algebra for V^+, B^+.

As an example, let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, and let B_n be the standard symmetric bilinear form on $\mathbb{F}^n$. Set $\mathscr{A}_0(\mathbb{F}) = \mathbb{F}$, and define $\mathscr{A}_n(\mathbb{F})$ and ν_n inductively by the above procedure. Then $\mathscr{A}_n(\mathbb{F})$, ν_n is a universal Clifford algebra for

$\mathbb{F}^n$, B_n. When $\mathbb{F} = \mathbb{R}$, the first step in this procedure yields $\mathbb{C}$, viewed as a real algebra, and the principal automorphism is complex conjugation. The second step constructs the usual quaternions, but note that the principal automorphism is not quaternionic conjugation because it reverses products.

The elements in the algebras $\mathscr{A}_n(\mathbb{F})$ may ultimately be identified with $2^n \times 2^n$ matrices, and these tend to be large—in fact, larger than necessary. Indeed, there are smaller realizations that more clearly reveal the essential properties of the algebras and are useful for other matters.

To describe one such realization, we first remark that for many applications to analysis and to representation theory, the most important case is that in which V is a real vector space of dimension n and B is an inner product. Assuming this and setting

$$\langle n \rangle = [n/2] \tag{6}$$

the greatest integer in $n/2$, one may then, by [10], construct a complex inner product space S (a *spin space*) of dimension $2^{\langle n \rangle}$ and a linear map γ (a *spinor* map) from V to an irreducible space of skew adjoint operators on S that satisfies (1). Given this, let $\mathscr{A}$ be the real subalgebra and $\mathscr{C}$ the complex subalgebra of $\mathrm{End}_{\mathbb{C}}(S)$ generated by $\gamma(V)$ and the identity operator on S. Then $\mathscr{A}$ and $\mathscr{C}$ are selfadjoint algebras of operators on S; $\mathscr{A}$, ν is a Clifford algebra for V, B; the algebra $\mathscr{C} = \mathscr{A} + i\mathscr{A}$; and the commutant of $\mathscr{C}$ consists of scalar multiples of the identity on S. Therefore, by the double commutant theorem, $\mathscr{C} = \mathrm{End}_{\mathbb{C}}(S)$ and so has dimension $2^{2\langle n \rangle}$. It follows that the real algebra $\mathscr{A}$ has dimension $2^{2\langle n \rangle}$. If the linear extension of γ and the bilinear extension of B to $V_{\mathbb{C}}$ are again denoted γ, B, then $\mathscr{C}$, γ is a Clifford algebra for $V_{\mathbb{C}}$, B. These algebras are universal, i.e., have dimension 2^n iff n is even.

This construction also yields Clifford algebras of operators for nonsingular indefinite forms, for suppose F is a nondegenerate symmetric bilinear form on V. Then there is a linear map $E : V_{\mathbb{C}} \to V_{\mathbb{C}}$ such that

$$F(x, y) = B(Ex, Ey)$$

for all $x, y \in V$. It follows that

$$\gamma(Ex)^2 = -B(Ex, Ex) = -F(x, x)$$

for all $x \in V$. Let $\mathscr{A}_F$ be the real subalgebra of $\mathscr{C}$ generated by $\gamma(E(V))$ and the identity operator on S. Then $\mathscr{A}_F$ is a Clifford algebra for V, F with embedding $\gamma \circ E$.

Now let τ be the trace function on $\mathrm{End}_{\mathbb{C}}(S)$ normalized so that $\tau(1) = 1$. Then $\mathrm{End}_{\mathbb{C}}(S)$ equipped with τ is a complex noncommutative probability algebra and a complex Hilbert algebra when endowed with the inner product

$$(X|Y) = \tau(XY^*). \tag{7}$$

It is clear that the same is true for the Clifford algebra $\mathscr{C}$, γ. It is also true, but not so obvious, that τ is real valued on $\mathscr{A}$ and hence that $\mathscr{A}$ equipped with τ is a real noncommutative probability algebra and a real Hilbert algebra relative to the inner product given by (7). For $x \in V$

$$\gamma(x)^2 = -|x|^2 = \tau(\gamma(x)^2) = -\tau(\gamma(x)\gamma(x)^*) = -|\gamma(x)|^2.$$

Thus, condition (1) implies, in the present context, that γ is a unitary map of V into $\mathscr{A}$.

An elementary inductive construction for the spinor maps γ may be based on the following result.

Lemma. *Let S be a complex inner product space of dimension $2^{\langle n\rangle}$, τ the normalized trace function on $\mathrm{End}_{\mathbb{C}}(S)$, and L an irreducible n-dimensional real subspace of $\mathrm{End}_{\mathbb{C}}(S)$ consisting of skew adjoint operators Y such that*

$$Y^2 = \tau(Y^2) = -|Y|^2.$$

Let M be the set of all operators on $S \oplus S$ that are specified, in the usual way, by matrices of the form

$$\begin{pmatrix} Y & x_1 + ix_2 \\ -x_1 + ix_2 & -Y \end{pmatrix} \tag{8}$$

where $Y \in L$ and $x_1, x_2 \in \mathbb{R}$ are identified with multiples of the identity on S. Then M is an irreducible real subspace of $\mathrm{End}_{\mathbb{C}}(S \oplus S)$ of dimension $n+2$ consisting of skew adjoint operators $Z = Z(x, Y)$ such that

$$Z^2 = -|Z|^2. \tag{9}$$

If Z is the operator defined by the matrix in (8), then (9) follows from the fact that

$$\begin{pmatrix} Y & x_1 + ix_2 \\ -x_1 + ix_2 & -Y \end{pmatrix}^2 = \begin{pmatrix} -x_1^2 - x_2^2 - |Y|^2 & 0 \\ 0 & -x_1^2 - x_2^2 - |Y|^2 \end{pmatrix}.$$

To see that M is an irreducible set, let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be the matrix of an operator that commutes with M; it must then commute with the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix}.$$

From this it follows that $A = D$, $B = -C$; that $B = 0$; and that $AY = YA$ for all $Y \in L$. Since L is an irreducible set, this implies A is a scalar multiple of the identity. Thus, M is also an irreducible set. The other assertions in the lemma are more or less evident and, in any case, easily established.

Two examples are relevant. If $n = 1$, we may take $S = \mathbb{C}$ and $L = \{(iy) : y \in \mathbb{R}\}$. If $n = 2$, we set $S = \mathbb{C}^2$ and

$$L = \left\{ \begin{pmatrix} iy_1 & y_2 \\ -y_2 & -iy_1 \end{pmatrix} : y_1, y_2 \in \mathbb{R} \right\}.$$

At this point, one may use the lemma and induction to construct spinor maps for every dimension.

The spinor map γ may also be used to define an explicit *spin* representation of the Lie algebra of skew adjoint operators on V on the spin space S [10]. But here it is inappropriate to devote more time and space to a development of the spin representation along these lines. Moreover, the authors treat this and related questions in great detail by other methods.

The book by Gilbert and Murray has five chapters, the first of which is purely algebraic. In Chapter 2 a multidimensional extension of classical Hardy space theory is given for Lipschitz domains in which holomorphic functions are replaced by functions in the kernel of the Dirac operator. In Chapter 3 some basic representation theory is presented and explicit spinor representations are studied. In particular, there is an abundance of interesting material, much of which is not well known and some of which is new, on the polynomial algebra on a space of rectangular matrices. Chapter 4 concerns constant coefficient operators of Dirac type. These are linear first-order elliptic operators obtained when one restricts the euclidian Dirac operator to appropriate subspaces of the Clifford module on which it acts. The generalized Cauchy-Riemann operators, first studied by Stein-Weiss [11], are examples; and this chapter presents a most welcome unified treatment of this very appealing topic. In the final chapter, Clifford algebras and Dirac operators are studied in the context of Riemannian manifolds. Here it is necessary to consider the influence of curvature and other geometric complications. The basic ideas are well illustrated by a section devoted to a careful discussion of the invariance properties of the Dirac operator on hyperbolic space. There is also a complementary section on spherical principal series representations of $SO(n, 1)$. The book concludes with a proof of the local Atiyah-Singer index theorem for Dirac operators [12].

Clifford algebras and Dirac operators in harmonic analysis contains a more-than-ample allotment of ideas and techniques that should be of great interest to a wide variety of analysts. The material itself is an attractive blend of algebra, analysis, and geometry. It is not particularly easy, but on the other hand, it is not impossibly difficult; and serious readers of this book should find it highly rewarding.

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