BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 2, October 1994

0273-0979/94 $1.00 + $.25 per page

Harmonic analysis and represention theory for groups acting on homogeneous trees, by Alessandro Figà-Talamanca and Claudio Nebbia. London Mathematical Society Lecture Notes Series, vol. 162, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, and Sydney, 1991, ix + 151 pp., $29.95. ISBN 0-521-42444-5

Harmonic analysis usually begins with a group but, sometimes, conceptually, more appropriately starts with a geometrical object attached to the group such as the symmetric space in the case of a semisimple Lie group. Rank 1 symmetric spaces are by far the easiest with which to deal. One avoids some very serious complications. Among the rank 1 spaces, the real hyperbolic $(m+1)$-space, $m \geq 1$, is especially simple. This space corresponds to the Lie algebra $\mathfrak{so}(1, m+1)$, which is sometimes conveniently written as $\mathfrak{sl}(2, \mathfrak{F})$ where the "field" $\mathfrak{F}$ is only required to be a real Hilbert space of dimension m. The cases of the reals, complex numbers, or quaternions correspond to $m = 1$, $m = 2$, and $m = 4$ respectively, but one does not really need the field structure. The symmetric space may be regarded as the cone, $\mathfrak{X}$, of positive definite matrices

$$\begin{pmatrix} a & u^* \\ u & d \end{pmatrix}, \qquad a, d > 0, u \in \mathfrak{F}; ad - \|u\|^2 > 0,$$

reduced by identifying two such matrices which are scalar multiples of one another. (Here u^* stands for another copy of u, with the suggestion that if multiplication were really defined, then we would have $u^*u = \|u\|^2 = uu^*$.) The Lie group $\mathbf{PL}^+(2, \mathfrak{F})$ which is the connected centre-free group with Lie algebra $\mathfrak{sl}(2, \mathfrak{F})$ may be defined as the component of the identity in the group of automorphisms of the identified positive-definite matrices. In considering these examples, one should not be restricted to the case $\mathfrak{F} = \mathbb{R}$ which has some special properties which are not relevant for the general case. The harmonic analysis here is simplified by considering all m at the same time.

The boundary, Ω, of the space of positive definite matrices (with the given equivalence relation) may be regarded as the space of matrices

$$\begin{pmatrix} a & u^* \\ u & d \end{pmatrix}, \qquad a, d \geq 0, u \in \mathfrak{F}; ad - \|u\|^2 = 0,$$

excluding $a = 0 = d$, with the same equivalence relation. It is easy to see that this space is diffeomorphic to S^m. It makes sense, however, to identify the space with the *projective line*, $\mathrm{Proj}(1, \mathfrak{F})$.

One can generalize to the Lie algebras $\mathfrak{sl}(n, \mathfrak{F})$ with $n > 2$ if $\mathfrak{F}$ is one of the classical skew fields or, when $n = 3$, the Cayley numbers. The symmetric space is again the obvious space of positive-definite matrices with the same equivalence relation.

There is a related algebraic topic which arises for the groups $\mathbf{PL}(n, \mathfrak{F})$ where $\mathfrak{F}$ is a commutative field of characteristic $\neq 2$. Consider the space of $n \times n$ symmetric matrices Y with coefficients in $\mathfrak{F}$. The action of $\mathbf{GL}(n, \mathfrak{F})$ is given by $Y \mapsto AYA^*$ where A^* is the ordinary transpose. We identify Y with αY if $\alpha \in \mathfrak{F}_*^2$, i.e., α is a nontrivial square. Let $\mathfrak{Y}$ be the space of invertible symmetric matrices modulo this equivalence relation. The orbits of $\mathbf{PL}(n, \mathfrak{F})$ in $\mathfrak{Y}$ give the rational equivalence classes of nondegenerate quadratic forms with coefficients in $\mathfrak{F}$. Of particular interest is the case in which $\mathfrak{F}$ is the field of quotients of a ring of integers $\mathfrak{O}$. We then have the question of determining the integral equivalence classes of quadratic forms. Notice that the scaling equivalence allows us to assume that symmetric matrices have integral entries. We may also restrict ourselves to elements of $\mathbf{GL}(n, \mathfrak{F})$ with integral entries when examining the orbits in $\mathfrak{Y}$. For integral equivalence what enters is $\mathbf{GL}(n, \mathfrak{O})$, the subgroup of $\mathbf{GL}(n, \mathfrak{F})$ constituted by those matrices that together with their inverses have entries in $\mathfrak{O}$. The free integral lattrices with n generators may be viewed as the homogeneous space $\mathbf{GL}(n, \mathfrak{F})/\mathbf{GL}(n, \mathfrak{O})$. Given the identifications we have been using, it is better to replace this by

$$\mathfrak{X} = \mathbf{GL}(n, \mathfrak{F})/(\mathfrak{F}_* \mathbf{GL}(n, \mathfrak{O}))$$

where $\mathfrak{F}_*$ is viewed as the centre of $\mathbf{GL}(n, \mathfrak{F})$. The points of $\mathfrak{X}$ are very closely related to the integral equivalence classes of quadratic forms.

In the book under review the subject is, basically, the space $\mathfrak{X}$ in the case $n = 2$ and $\mathfrak{F}$ a local field, i.e., a totally disconnected locally compact field. Here the integers have a unique maximal ideal $\mathfrak{P}$, and it is principal. We write π for a fixed choice of a generator of $\mathfrak{P}$. Consider the point $o \in \mathfrak{X}$ represented by the identity $I \in \mathbf{GL}(2, \mathfrak{F})$. Say that $y \in \mathfrak{X}$ is *adjacent* to o if it is represented by an integral matrix with determinant in $\mathfrak{P}\backslash\mathfrak{P}^2$. A little thought will show that the distinct adjacent points are represented by the matrices

$$\begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -\varepsilon & 1 \\ \pi - \varepsilon^2 & \varepsilon \end{pmatrix}$$

where ε runs through the representatives of $\mathfrak{O}/\mathfrak{P}$. For the second matrix, if one adds ε times column 2 to column 1, one gets the more obvious form $\left(\begin{smallmatrix} 1 & 0 \\ \varepsilon & \pi \end{smallmatrix}\right)$; the form chosen exhibits matrices whose square is πI so that they are involutions of $\mathfrak{X}$. Extend the notion of *adjacency* by homogeneity. We obtain a graph by taking the points of $\mathfrak{X}$ as vertices and pairs of adjacent points constituting the edges. One can verify that this graph is a tree. Each vertex is incident with $q + 1$ edges where q is the number of elements in the finite field $\mathfrak{O}/\mathfrak{P}$.

The point of view of the authors is to consider a general homogeneous tree $\mathfrak{X}$ of degree $q + 1$ where q need not be a prime power. The basic group under consideration is $\mathrm{Aut}(\mathfrak{X})$, the full group of automorphisms of the tree with the topology given by choosing a basic system of neighborhoods of the identity to be the stabilizers of finite subsets of $\mathfrak{X}$. This topology is locally compact Hausdorff. Applications to harmonic analysis on $\mathbf{PL}(2, \mathfrak{F})$ with $\mathfrak{F}$ a local field will follow by considering these as closed subgroups of $\mathrm{Aut}(\mathfrak{X})$ which act transitively on $\mathfrak{X}$. (The subgroups $\mathbf{PSL}(2, \mathfrak{F}) \subset \mathbf{PL}(2, \mathfrak{F})$ do not act transitively on $\mathfrak{X}$ but, rather, have two orbits, each constituted by the points at even distance from each other.)

A *geodesic ray* emanating from a point is a path without self-intersections starting at that point. The distinct geodesic rays emanating from a given point form the boundary Ω; this is obviously independent of the starting point. In the case of the tree arising from a local field, Ω can be identified with the projective line. In all cases there is a natural topology in which Ω and $\mathfrak{X} \cup \Omega$ are compact. The topology of $\mathfrak{X}$ itself is discrete. This gives a technical simplification of what analysis one needs to do on a rank 1 symmetric space; harmonic functions are those whose value at a point is the average of the values at the $q+1$ nearest neighbors rather than solutions of a differential equation. The probabilistic interpretation uses a simple random walk which everyone can understand without getting into measure-theoretic complexities.

There are some notable differences between the classical semisimple Lie groups of noncompact type and groups acting on homogeneous trees. In the group of isometries, $\mathbf{G} = \mathrm{Aut}(\mathfrak{X})$, of a classical symmetric space $\mathfrak{X}$, the maximal compact subgroups of $\mathbf{G}$ are precisely the stabilizers of the points $p \in \mathfrak{X}$. For the case of homogeneous trees there are additional maximal compact subgroups of $\mathbf{G} = \mathrm{Aut}(\mathfrak{X})$, namely, the subgroups which stabilize an edge. Once again, in the classical case $\mathrm{Aut}(\mathfrak{X})$ has only a finite number of components, and the component of the identity has no proper closed semisimple subgroup which acts transitively on $\mathfrak{X}$. By contrast, $\mathrm{Aut}(\mathfrak{X})$ is a much bigger group than $\mathbf{PL}(2, \mathfrak{F})$. There are important differences in the harmonic analysis of the various semisimple groups $\mathbf{G}$ acting faithfully and transitively on a homogeneous tree $\mathfrak{X}$.

One of the early results cited in the book is that a closed subgroup of $\mathrm{Aut}(\mathfrak{X})$ is amenable iff (i) it is compact, (ii) it fixes a point of the boundary, or (iii) it stabilizes a geodesic. By contrast, a nonamenable closed subgroup contains a discrete free group on two generators. This brings up a comparison with [F-TP]. Harmonic analysis on free groups and harmonic analysis on groups acting on homogeneous trees have a great deal in common, and the topics of the present volume are similar to those of [F-TP]: analysis of the Laplace operator and the study of unitary representations. The book at hand is divided into three chapters of approximately equal length, with the first devoted to generalities and the next two concentrating on the Laplace operator, spherical functions, and representation theory. Chapter III begins with a classification of irreducible unitary representations as *spherical, special*, and *cuspidal.* The last are quite important for $\mathrm{Aut}(\mathfrak{X})$, but the results obtained do not apply to $\mathbf{PL}(2, \mathfrak{F})$. The book concludes with an appendix describing the connection between $\mathbf{PL}(2, \mathfrak{F})$ and trees.

A vexing problem with Lie theory is that there is an enormous amount of preliminary material which one has to learn. Attempts to treat the subject as if $\mathbf{SL}(2, \mathbb{R})$ were a typical Lie group are absurd. One can give concrete interpretations to much of what arises when dealing with classical groups, but at some point one must come to grips with the abstract theory if one wishes to go beyond real rank 1. The rank 1 case, especially the Lie groups corresponding to the real hyperbolic spaces, allow one to do a lot "by hand". The Laplace-Beltrami operator on the symmetric space applied to functions with the usual symmetry properties reduces to an ordinary differential operator rather than a partial differential operator. By doing harmonic analysis on homogeneous trees where the Laplace operator is discrete, the authors have avoided differentiation. For the reader interested in learning the basic ideas of modern representation theory, this is an advantage. The rank 1 Lie groups with their associated ordinary differential equations are easier to handle in some respects, but the discrete symmetric spaces bring up problems of Lie groups over local fields which are very important in number theory. As indicated, one drawback of dealing with homogeneous trees is that one does not see what to do for $\mathbf{SL}(3, \mathfrak{F})$.

Figà-Talamanca and Nebbia have written an interesting book which is reasonably accessible to those with a knowledge of abstract harmonic analysis. The contents of the present work go beyond that of the earlier book by Figà-Talamanca and Picardello [F-TP], especially in view of the applications to number theory as mentioned above. To this reviewer, the earlier book is more exciting to read. No doubt this is due to the fact that in 1983 the subject was still in the developmental stage. Harmonic analysis on trees goes back to ideas developed in the late 1960s and early 1970s by Bruhat and Tits, Serre, and Cartier, among others; [BT, S, C] give a sampling of the pioneering work (dates of publication are misleading; [S] is the bibliographically accessible version of lectures given by Serre in 1967/68). By 1991 the subject had reached the mature stage. While the book under review does not study semihomogeneous trees and does not give all the details for harmonic analysis on free groups, it is a reasonably definitive work on its subject. That subject is really $\mathrm{Aut}(\mathfrak{X})$; many results remain valid for interesting subgroups, but, as the authors observe, the treatment of cuspidal representations in this book does not apply to $\mathbf{PL}(2, \mathfrak{F})$. Thus there remain interesting and important questions about harmonic analysis on certain groups which act on homogeneous trees. For these questions the tree structure would not seem to be the essential ingredient. For the more basic material where the tree structure provides a good intuitive guide, this work of Figà-Talamanca and Nebbia is highly recommended.

References

[BT] F. Bruhat et J. Tits, *Groupes réductifs sur un corps local,* Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–252.

[C] P. Cartier, *Fonctions harmoniques sur un arbre,* Sympos. Math., vol. 9, Academic Press, London, 1972, pp. 203–270.

[F-TP] A. Figà-Talamanca and M. A. Picardello, *Harmonic analysis on free groups,* Lecture Notes in Pure and Appl. Math., vol. 87, Marcel Dekker, New York, 1983.

[S] J.-P. Serre, *Arbres, amalgams,* $\mathbf{SL}_2$, Astérisque **46** (1977), 1–189.

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