instructive, therefore, also to give the headings of the fourteen chapters that constitute volumes I and II:

Linear spaces
Basics of Hilbert space and linear operators
Banach algebras
Elementary C^*-algebra theory
Elementary von Neumann algebra theory
Comparison theory of projections
Normal states and unitary equivalence of von Neumann algebras
The trace
Algebra and commutant
Special representations of C^*-algebras
Tensor products
Approximation by matrix algebras
Crossed products
Direct integrals and decompositions

Many years ago Paul Halmos published a delightful book called *A Hilbert space problem book* that presented the elementary theory of operators in a series of problems (with hints and solutions). Appealing though this approach may be, it will probably not work in a highly technical field like operator algebras, where the teacher must step in from time to time to tell the student about heavy machinery that has to be developed before further progress can be made. Yet Halmos's dictum stands: *The only way to learn mathematics is to do mathematics.*

The completed four-volume treatise by Kadison and Ringrose seems to me to utilize the best of both methods: The fundamentals are explained as text to be read. The numerous exercises are inserted to challenge the curiosity, to develop "hands-on" skills, and to give a glimpse of wider spaces. Now the solutions, as in Halmos's book, appear at the end as the logical conclusion. The authors have erected a monument in mathematics in the tradition of Courant-Hilbert, Dunford-Schwartz, Hewitt-Ross, and Reed-Simon.

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An introduction to Γ-convergence, by Gianni Dal Maso, Birkhäuser, Boston, 1992, xiv+337 pp., $69.50. ISBN 0-8176-3679-X

Not long ago, a colleague at Courant Institute asked me: "Do you know a good reference on Γ-convergence and 'all that stuff'?" I realized that was not an easy question to answer. Although the topic has existed for over thirty years, I could not think of a single book or set of lecture notes that covered reasonably well the theory of Γ-convergence and also gave an idea of its main applications (which I interpreted as the "all that stuff" part of the question). This was before I became aware of Dal Maso's book. In spite of its modest claim, *An introduction to Γ-convergence* is much more than just that. I found a complete discussion of the subject that originated in the now-famous work of De Giorgi [6] on calculus of variations. Γ-convergence has since permeated many aspects of analysis, particularly in the area of partial differential equations. Moreover, Γ-convergence was a crucial element in the development of some of the more formal aspects of homogenization theory, in the analysis of Ginsburg-Landau equations for modeling phase transitions, and in many other areas of applied mathematics.

Roughly speaking, Γ-convergence is the correct topological notion for the study of the convergence of functionals in the calculus of variations. To give a "flavor" of how some of these problems come about and of the role that Γ-convergence plays in their solution, let me describe a few examples of Γ-convergence "at work".

1. A PHASE-FIELD EQUATION

Modica [9], Sternberg [14], and Kohn and Sternberg [7], among others, considered the following variational problem: Given a domain Ω in R^d, minimize the functional

$$F_\varepsilon[u] = \varepsilon \int_\Omega |\nabla u(x)|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W[u(x)]\, dx, \tag{1}$$

where

$$W[u] = \frac{u^4}{4} - \frac{u^2}{2} \tag{2}$$

is a "double-well' potential with minima at $u = \pm 1$ and ε is a small parameter, subject to the constraint

$$\int_\Omega u(x)\, dx = c = \text{const}. \tag{3}$$

The question addressed by these authors was to characterize the asymptotic behavior of the minimizers of (1) as $\varepsilon \to 0$. The problem itself is motivated by the Landau-Ginsburg theory of phase transitions, in which the function $u(x)$ is an order-parameter representing the state of the system at the point x (see Caginalp [3]). Here, $u(x) = +1$ and $u(x) = -1$ represent different phases, and W is often referred to as a *bistable* potential. In the absence of the gradient term in (1), minimizers are expected to satisfy $u(x) = \pm 1$ almost everywhere in the domain Ω in such a way that the integral constraint (3) is also satisfied. This argument also applies to the minimizers of (1) for $\varepsilon \ll 1$, since the gradient term becomes negligible compared to the W term in the variational integral.

A subtle and interesting question is to understand the shape of the interface between regions inside Ω where the minimizer satisfies $u(x) \approx 1$ and $u(x) \approx -1$, for $\varepsilon \ll 1$. The answer is provided in

Theorem 1. *Let $u_\varepsilon(x)$ be a sequence of minimizers of* (1) *satisfying the integral constraint* (3). *Then $\{u_\varepsilon\}$ has a subsequence which converges in $L^1(\Omega)$ to $u_0(x)$, where this function is a minimizer of the functional*

$$F_0[u] = \text{Perimeter}\{x \in \operatorname{int}\Omega \colon u(x) = 1\},$$

over the class of functions satisfying $u(x) = \pm 1$ *almost everywhere and* (3).

Thus, the limit of the sequence of minimizers is associated with an *isoperimetric problem*: find a partition of Ω into two subsets Ω_+ and Ω_- such that $|\Omega_+| - |\Omega_-| = c$ and such that the common boundary of the two sets has the smallest possible $(d-1)$-measure.

2. Brinkman approximation for diffusion in a medium with absorbing traps

These results were obtained by Rauch and Taylor [12], Papanicolaou and Varadhan [11], Cioranescu and Murat [4], Baxter and Jain [1], and Dal Maso and Mosco [5]. Consider the problem of steady-state diffusion in a domain Ω in R^d in which the production of diffusing particles from a spatially distributed source $f(x)$ is balanced by absorption through a system of many small spherical "traps" in the medium. Mathematically, this can be modeled as a Dirichlet problem

$$\begin{cases} -\Delta u_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ u_\varepsilon = 0, & x \notin \Omega_\varepsilon, \end{cases}$$

where u_ε is the particle density and $\Omega_\varepsilon = \Omega - \{\bigcup_i B_i(r_\varepsilon)\}$. Here, $B_i(r_\varepsilon) = \{x \in R^d \colon |x - \varepsilon z_i| \le r_\varepsilon\}$, $z_i \in Z^d$, so that $\{B_i(r_\varepsilon)\}$ is a collection of spheres of radius r_ε with centers on a cubic lattice of mesh ε representing the "traps". Let $E_\varepsilon = \Omega \cap \{\bigcup B_i(r_\varepsilon)\}$. The question of interest is, once again, to describe the behavior of $u_\varepsilon(x)$ as $\varepsilon \to 0$. I will focus here on the particularly interesting case where $r_\varepsilon = a\varepsilon^{d/(d-2)}$, which corresponds to having a uniformly bounded capacity of the set of absorbers E_ε.

Let $\tilde{u}_\varepsilon(x) \in H_0^1(\Omega)$ be the extension of $u_\varepsilon(x)$, $x \in \Omega_\varepsilon$, which is equal to zero inside the traps, i.e.,

$$\tilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x \in \Omega_\varepsilon, \\ 0, & x \in E_\varepsilon. \end{cases}$$

Then, it is easy to see that $\tilde{u}_\varepsilon$ is the minimizer of the functional

$$F_\varepsilon[u] = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega I_\varepsilon(x)u(x)\,dx + \int_\Omega f(x)u(x)\,dx,$$

among all $u \in H_0^1(\Omega)$, where

$$I_\varepsilon(x) = \begin{cases} +\infty & \text{if } x \in E_\varepsilon, \\ 0 & \text{if } x \notin E_\varepsilon. \end{cases}$$

Thus, our problem is again equivalent to finding the asymptotic behavior of the minimizers of a sequence of problems in the calculus of variations. In this case the solution is given by

Theorem 2. *The sequence* $\{\tilde{u}_\varepsilon(x)\}$ *converges in* $H_0^1(\Omega)$ *to* $u_0(x)$, *where this function is the minimizer of the functional*

$$F_0[u] = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega 4\pi a u(x)\,dx + \int_\Omega f(x)u(x)\,dx. \tag{4}$$

Notice that the Euler-Lagrange equation for (4) is

$$\begin{cases} -\Delta u_0(x) + 4\pi a u_0(x) = f(x) & \text{in } \Omega, \\ u_0(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

The coefficient $4\pi a$ can be interpreted as the overall capacity (in the potential-theoretic sense) of the system of traps. Thus, macroscopically, the effect produced by a "fine-grained" system of absorbers with bounded capacity is equivalent to "continuously" removing particles from the system at a constant rate proportional to the capacity. This result is known in the chemical engineering literature as the Brinkman approximation.

3. Definition of Γ-convergence

The common theme of these two examples is that the behavior of a *sequence of minimizers of variational problems* is studied: for each $\varepsilon > 0$, u_ε is the minimizer of $F_\varepsilon[u]$, $u \in H$, where H is a function space or a closed subset of a function space. The limit of the sequence $\{u_\varepsilon\}$ is characterized in terms of a "limiting" variational problem $\min_{u\in H} F_0[u]$. Γ-convergence is a topology on the space of functionals which is tailored to reflect the property of "convergence of minimizers".

Definition. Let $\{F_\varepsilon\}_{\varepsilon>0}$ and F_0 be functionals defined on a topological vector space H. We say that $\{F_\varepsilon\}$ Γ-converges to F_0 ($\Gamma - \lim_{\varepsilon\to 0} = F_0$) if and only if

(i)

$$F_0[u] \leq \liminf_{\varepsilon\to 0} F_\varepsilon[u] \tag{5}$$

for all u in H; and

(ii) for all $u \in H$, there exists a sequence $\{u_\varepsilon\}$ converging to u in H such that

$$\limsup_{\varepsilon\to 0} F_\varepsilon[u_\varepsilon] \leq F_0[u]. \tag{6}$$

Γ-convergence defines a topology on the set of (nonlinear) functionals on H. In Dal Maso's book, the relation between Γ-convergence and other topologies is elucidated. One of the fundamental properties of Γ-convergence is that (5) and (6) are essentially necessary and sufficient conditions for the aforementioned property of "convergence of minimizers" to hold.

Γ-convergence is the correct mathematical framework to formulate rigorously a variety of problems coming mostly from physics. In these problems, the quantities of interest satisfy variational principles like the ones mentioned above. The dependence of the functional on a small parameter (ε in the above examples) reflects the fact that the problem in question has two or more relevant physical scales (measuring distance, time, mass, etc.). The small parameter represents typically the ratio of these scales. In many cases, the existence of a Γ-limit is associated with a simple description of the physical phenomenon, as we saw in the two examples.

4. Homogenization theory

The standard reference for this subject is Bensoussan, Lions, and Papanicolaou [2]. I will discuss here the simplest situation, corresponding to the macroscopic behavior of a composite material, say, a thermal insulator. Consider a domain Ω in R^3 and the Dirichlet problem

$$\begin{cases} \nabla \cdot \kappa(\frac{x}{\varepsilon}) \cdot \nabla T_\varepsilon(x) = 0, & x \in \Omega, \\ T_\varepsilon(x) = g(x), & x \in \partial\Omega, \end{cases}$$

where $g(x)$ can be interpreted as a steady-state temperature on the boundary of a material occupying the region Ω. The function $\kappa(\cdot)$ is periodic with period 1 and satisfies $\kappa_1 \le \kappa(x) \le \kappa_2$, where κ_1 and κ_2 are positive constants. For instance, we can consider the case in which $\kappa(x)$ takes only two values κ_1 and κ_2, which corresponds to a *two-phase composite* with constituents arranged in a periodic manner. The parameter ε represents the ratio between the "macroscopic" scale (the diameter of the domain Ω or the scale of variation of the function g) and the "microscopic" scale (the typical scale on which the thermal conductivity varies). Usually ε is quite small, and the problem of interest is to understand the asymptotic limit $\varepsilon \to 0$ of the function T_ε and its gradient ∇T_ε. Recall from elementary physics that, from Fourier's Law,

$$Q_\varepsilon(x) = -\kappa\left(\frac{x}{\varepsilon}\right)\nabla T_\varepsilon(x)$$

represents the local *thermal flux* at the point x. As $\varepsilon \to 0$, the composite is expected to behave like a homogeneous continuum with constant thermal conductivity κ^*. Mathematically, this means that

$$\lim_{\varepsilon \to 0} T_\varepsilon(x) = T_0(x)$$

and

$$\lim_{\varepsilon \to 0} Q_\varepsilon(x) = -\kappa^* \nabla T_0(x),$$

which can be regarded as an "effective" or "homogenized" Fourier's Law. To study the existence of a homogenized conductivity and to investigate its properties, one can set up the problem as the study of the Γ-limit of the sequence of quadratic functionals

$$F_\varepsilon[T] = \int_\Omega \kappa\left(\frac{x}{\varepsilon}\right)\nabla T(x)\cdot\nabla T(x)\,dx,$$

for T in $H_0^1(\Omega)$ (endowed with the weak topology). The Γ-limit of $\{F_\varepsilon\}$ is characterized in

Theorem 3.

$$\Gamma - \lim_{\varepsilon \to 0} F_\varepsilon = F_0,$$

where

$$F_0[T] = \int_\Omega \kappa^* \nabla T(x)\cdot\nabla T(x)\,dx. \tag{7}$$

The homogenized conductivity tensor κ^* *admits the following characterization: let* $Q = [0, 1] \times [0, 1] \times [0, 1]$. *Then, for all* $\theta \in R^3$,

$$\theta \cdot \kappa^* \cdot \theta = \int_Q \kappa(y)|\theta + \nabla\chi(y)|^2\,dy,$$

where $\chi(y)\colon R^3 \to R$ *is a Q-periodic solution of the PDE*

$$\nabla\cdot[\kappa(y)(\theta + \nabla\chi(y))] = 0.$$

This theorem, the proof of which can be found in [2, 13], admits numerous generalizations to other elliptic systems with variational integrands, including elasticity theory, Stokes flow, nonlinear constitutive laws, etc.

5. Dal Maso's book

As the reader probably suspects, the three examples given here require different, specific mathematical tools for their solution. In fact, it is possible to formulate and solve these problems in applied mathematics in a "concrete" way, without recourse to the abstract theory of Γ-convergence. Nevertheless, much like other areas of mathematics such as, say, the theory of distributions, Γ-convergence provides us with a unified framework for thinking about many different problems. In this reviewer's opinion, Γ-convergence represents, more than anything else, a *general, conceptual theory for studying asymptotic problems in the calculus of variations.*

In his book, Dal Maso makes great efforts to develop a complete theory of convergence of functionals. A useful study is made, comparing Γ-convergence to other topologies on functionals defined on topological vector spaces, such as the Kuratowski topology. The conditions which ensure *compactness* of a sequence of functionals under the topology induced by Γ-convergence are studied in detail. (This issue has found some interesting applications in the theory of optimal design of composites; see for instance Kohn and Strang [8] and Murat [10].) As an "application", the book studies the theory of Γ-convergence of sequences of classical functionals in the calculus of variations of the form

$$F[u] = \int_\Omega f(x, u(x), \nabla u(x))\, dx,$$

where $f(x, u, p)$ satisfies appropriate regularity and growth conditions. The case of quadratic functionals, i.e., $f(x, u, p) = \sum_{i,j=1}^d a_{ij}(x)p_i p_j$, mentioned in Theorem 3, also gets a lot of attention. In this case, it is known that Γ-convergence is "morally equivalent" to the convergence of the resolvents of the associated elliptic differential operators. All this is discussed with great detail, and many interesting examples and counterexamples are given.

The reader interested in the applications of the theory will find that the book covers mostly applications to homogenization theory. This is done in the last two chapters as a brief but suggestive introduction to this vast subject.

Clearly, a separate book (or even several books) could be written on "applications" or, more precisely, on mathematical problems related to Γ-convergence. *An introduction to Γ-convergence* also presents what can be considered as a predecessor of such a book, in the form of a superb guide to the literature and a bibliography grouping about 1,000 papers on topics related to Γ-convergence according to the specific area of application. This is, without doubt, the most important bibliographical review on papers on homogenization theory, composite materials theory, singular perturbations, and related subjects compiled to date. It is fair to say that this bibliography is essentially complete up to 1991.

Due to its clear presentation of the subject and its view toward applications, this book will be a standard reference for years to come. *An introduction to Γ-convergence* will be particularly useful for graduate students and researchers in analysis and applied mathematics, who will add their own discoveries to the broad list of mathematical problems that can be understood via Γ-convergence.

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Dynamical systems VIII (*singularity theory* II), *classification and applications*, by V. I. Arnol'd, V. V. Goryunov, O. V. Lyashko, and V. A. Vasil'ev, editors. Encylopaedia of Mathematical Sciences, vol. 39, Springer-Verlag, New York, 1993, 235 pp., $89.00. ISBN 3-540-53376-1

Singularity theory is not a theory in the usual (axiomatic) sense. Indeed it is precisely its width, its vague boundaries, and its interaction with other branches of mathematics and science which makes it so attractive. In particular, the subject draws on algebraic and analytic geometry, commutative algebra, and differential analysis and has applications to differential and algebraic geometry, bifurcation theory, optics, and a wide range of other topics. This width then is encouraging, indeed exciting. But a nebulous nature can lead to identity crises, and I then find it useful to think of singularity theory as the direct descendant of differential calculus. It has, for example, the same concerns with Taylor series, and one can view much current research as natural extensions of problems with which our forefathers laboured and which were considered central. The calculus is the tool, par excellence, for studying physics, differential equations