attractive. The current work would certainly prove very useful to experts in the field. I would not, on the other hand, describe it as an introduction to the subject or a useful general reference for the interested reader. In particular, the last two chapters are rather technical (I certainly found them more difficult to read than the first three). There are of course no proofs (wouldn't my undergraduate class be pleased!), but proofs do go a long way towards explaining the reasons why results hold. In fact this really is a collection of (nicely written) survey articles whose topics lie within, or are adjacent to, singularity theory, and as such I can heartily recommend it.

Finally I feel that I must mention the price. This is a Springer volume of 235 pages weighing in at \$89. I am glad I had the book to review, as it is a valuable addition to my bookshelf. I doubt that I would have parted with \$89 for it. I wonder how much longer we academics are going to continue to create our own resource crises? Publishing is an activity we dominate: we write the papers/books, we do the refereeing, and we do the buying. And we still allow publishers to charge us exhorbitant prices for the privilege of having our work in our own libraries.

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Evolutionary integral equations and applications, by Jan Prüss, Monographs in Mathematics, vol. 87, Birkhäuser Verlag, Boston, MA, 1993, xxvi+366 pp., \$139.00. ISBN 3-7643-2876-2

This book deals with linear integral equations of the following type:

$$u(t) = \int_0^t A(t-s)u(s)\,ds + f(t), \qquad t \geq 0 \tag{1}$$

(and also with the corresponding equation on the line, see equation (3) below) where $u\colon [0, +\infty[\to X$ is a function with values in a Banach space X; $A(t)$, $t \geq 0$, are linear operators, generally unbounded; and $f\colon [0, +\infty[\to X$ is a given function.

There is an expansive literature regarding equation (1) when the space X is finite dimensional; see, for example, the monograph [3]. In infinite dimensions there are also several papers and, in addition, a monograph about nonlinear problems in viscoelasticity [6] but not, to our knowledge, a systematic treatise before the present book.

Equation (1) is studied by looking for a resolvent $S(t)$, $t \geq 0$, that is an operator-valued mapping such that (formally)

$$S(t) = \int_0^t A(t-s)S(s)\,ds + I, \qquad t \geq 0.$$

Once the resolvent is obtained, the solution to (1) is given by

$$u(t) = \frac{d}{dt}\int_0^t A(t-s)S(s)f(s)\,ds.$$

There are obviously several difficulties connected with this program, due to the lack of boundedness of operators $A(s)$. In the book, the case when $A(s) = K(s)A$, where $K(\cdot)$ is a scalar kernel, is first treated. Because it is obviously impossible to give a detailed description of all the results contained in the book, we quote, for instance (freely), the following generation result, which is a generalization of the classical Hille-Yosida Theorem for a strongly continuous semigroup of linear operators.

Theorem 1. *Let $A(t) = a(t)A$ with $A\colon D(A) \subset X \to X$ a closed, densely defined operator and a scalar Laplace transformable kernel, with $\int_0^{+\infty} e^{-\omega t}|a(t)|\,dt < +\infty$. Then there exists a resolvent $S(\cdot)$ for equation* (1), *if and only if, for $\operatorname{Re}\lambda > \omega$, the Laplace transform $\hat{a}(\lambda)$ is different from* 0, *$1/\hat{a}(\lambda)$ belongs to the resolvent set of A, and $H(\lambda) = 1/\lambda(I - \hat{a}(\lambda)A)$ fulfills the following estimates*:

$$\left\|\frac{d^nH}{d\lambda^n}\right\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}}, \qquad n = 1, 2, \ldots; \ \operatorname{Re}\lambda > \omega. \tag{2}$$

Obviously it is not easy to check the infinitely many estimates of (2) in general. However, this can be done for instance if one of the following statements holds

(i) X is a Hilbert space, A is selfadjoint nonnegative, and a is a positive kernel, that is, $\operatorname{Re}\hat{a}(\lambda) \geq 0$ whenever $\operatorname{Re}\lambda \geq 0$.

(ii) A generates an analytic semigroup, and $\hat{a}(\lambda)$ is extendable in a sector $S_{\omega,\theta} = \{\lambda \in \mathbf{C} : \arg|\lambda - \omega| < \theta\}$, $\theta > \pi/2$, of the complex plane, and

$$\|H(\lambda)\| \leq \frac{C}{|\lambda - \theta|}, \qquad \lambda \in S,$$

for some constant $C > 0$. This is the so-called *parabolic case*.

(iii) A generates a contraction semigroup and a is a completely positive kernel; see [1].

Moreover, an important tool to prove existence of resolvents for other problems (both parabolic and hyperbolic) is the so-called *subordination principle*; see [5].

Among the other arguments treated in the book, let us recall

(i) generalisation of generation theorems to nonscalar kernels,

(ii) a new variational approach to integrodifferential equations, and

(iii) maximal regularity results in the parabolic case.

The last part of the book is devoted to integrodifferential equations on the line

$$u(t) = \int_{-\infty}^{t} A(t-s)u(s)\,ds + f(t), \qquad t \geq 0. \tag{3}$$

Equation (3) is studied in homogeneous spaces, that is, spaces of functions on the real line where translations are bounded operators; this is a natural requirement due to the translation invariance of equation (3). Homogeneous spaces include: spaces of uniformly continuous and bounded functions, L^p functions, and periodic and almost periodic functions. The main tool used here is an integrability property of the resolvent $S(t)$ that is studied in different situations.

The prerequisites to read the book are essentially: basic functional analysis and semigroup theory. The method used can be seen as a generalisation to that theory. For instance, if $a(t) = 1$ and $f(t) = x$, equation (1) reduces to

$$u'(t) = Au(t), \qquad u(0) = x; \tag{4}$$

and if $a(t) = t$ and $f(t) = x + ty$, equation (1) reduces to

$$u''(t) = Au(t), \qquad u(0) = x, \qquad u'(0) = y.$$

It is possible, under suitable conditions and by an appropriate choice of the state spaces, to reduce a general equation (1) to an evolution equation of the form (4); see [4]. As pointed out by the author, this approach presents several disadvantages due to the arbitrariness of the choice of the state space, which looks somewhat artificial, and the lack of regularity properties of the previous system. For instance, starting from parabolic integrodifferential equations, one can find hyperbolic evolution equations. However, this second approach is very useful in the study of a linear quadratic control problem associated with problem (1) because it allows the use of dynamic programming; see [2]. The same happens if one is perturbing (1) by a white noise, since, in this form, the Markov property is lost.

In my opinion this book is written in a very clear and effective way. Moreover, several applications of the theory are extensively presented, with a careful description. In particular, we mention applications to: viscoelasticity, heat conduction in material with memory, thermo-viscoelasticity, and electrodynamics with memory.

In conclusion, the book is welcome, thermo-viscoelasticity and I am sure that it will be very useful for all scientists working in the field of linear integrodifferential equations.

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Wulff construction, A global shape from local interaction, by R. Dobrushin, R. Kotecky, and S. Shlosman. American Mathematical Society, Providence, RI, 1992, ix + 204 pp., $130.00. ISBN 0-8218-4563-2

What is the shape which has the least surface area for the volume it encloses? We all know the answer to that question—a round ball, in Euclidean space of any dimension. This is an *isoperimetric inequality*, a formula giving a lower bound on the $(n-1)$-dimensional area of the boundary ∂S of an n-dimensional region S in $\mathbf{R}^n$ (or more generally of a piece of n-dimensional minimal surface S in $\mathbf{R}^{m+n}$) in terms of the n-dimensional volume of S. The theorem not only gives a precise bound but says it is uniquely obtained by a single shape. See Osserman's review [O] for some of its history.

The Wulff construction answers the same question with area replaced by surface energy: *what is the shape which has the least surface energy for the volume it encloses*?

The construction itself can be stated succinctly: Given any function Φ from unit vectors in $\mathbf{R}^n$ to $\mathbf{R}$, the Wulff shape is

$$W_\Phi = \{x \in \mathbf{R}^n : x \cdot \mathbf{n} \leq \Phi(\mathbf{n}) \ \forall \mathbf{n} \in \mathbf{R}^n \text{ with } |\mathbf{n}| = 1\}.$$

The description of W_Φ used by materials scientists is very geometric, rather than formulaic, and goes approximately as follows: Plot the points $\Phi(\mathbf{n})\mathbf{n}$ for all unit vectors $\mathbf{n}$ (this is the "γ plot", since what is here called Φ—the terminology of geometric analysis, where Φ is a "parametric integrand"—is often called γ in the materials science literature; to confuse matters further, it is often called σ in the physics literature, and in this book it is τ). Now for each point on this plot, construct the plane perpendicular to the line from that point back to the origin and throw away everything beyond that plane. What is left after all those half spaces are discarded is W_Φ. See Figure 1.

In applications, $\Phi(\mathbf{n})$ is taken to be the surface free energy per unit area (a.k.a. surface tension) for a plane segment having oriented normal $\mathbf{n}$ separating one material from another. So, if one has a chunk S of one kind of stuff