3. G. Gripenberg, S. O. Londen, and O. J. Staffans, *Volterra integral and functional equations*, Cambridge Univ. Press, Cambridge, 1990.
4. R. K. Miller, *Linear Volterra integrodifferential equations as a semigroup*, Funkcial. Ekvac. **17** (1974), 39–55.
5. J. Prüss, *Linear hyperbolic Volterra equations of scalar type, semigroup theory and applications* (Ph. Clément, S. Invernizzi, E. Mitidieri, and I. Vrabie, eds.), Dekker, New York, 1989, pp. 367–384.
6. M. Renardy, W. J. Hrusa, and J. A. Nohel, *Mathematical problems in viscoelasticity*, Pitman Monographs Surveys Pure Appl. Math., vol. 34, Longman Sci. Tech., Harlow, Essex, 1988.

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Wulff construction, A global shape from local interaction, by R. Dobrushin, R. Kotecky, and S. Shlosman. American Mathematical Society, Providence, RI, 1992, ix + 204 pp., $130.00. ISBN 0-8218-4563-2

What is the shape which has the least surface area for the volume it encloses? We all know the answer to that question—a round ball, in Euclidean space of any dimension. This is an *isoperimetric inequality*, a formula giving a lower bound on the $(n-1)$-dimensional area of the boundary ∂S of an n-dimensional region S in $\mathbf{R}^n$ (or more generally of a piece of n-dimensional minimal surface S in $\mathbf{R}^{m+n}$) in terms of the n-dimensional volume of S. The theorem not only gives a precise bound but says it is uniquely obtained by a single shape. See Osserman's review [O] for some of its history.

The Wulff construction answers the same question with area replaced by surface energy: *what is the shape which has the least surface energy for the volume it encloses*?

The construction itself can be stated succinctly: Given any function Φ from unit vectors in $\mathbf{R}^n$ to $\mathbf{R}$, the Wulff shape is

$$W_\Phi = \{x \in \mathbf{R}^n : x \cdot \mathbf{n} \le \Phi(\mathbf{n}) \ \forall \mathbf{n} \in \mathbf{R}^n \text{ with } |\mathbf{n}| = 1\}.$$

The description of W_Φ used by materials scientists is very geometric, rather than formulaic, and goes approximately as follows: Plot the points $\Phi(\mathbf{n})\mathbf{n}$ for all unit vectors $\mathbf{n}$ (this is the "γ plot", since what is here called Φ—the terminology of geometric analysis, where Φ is a "parametric integrand"—is often called γ in the materials science literature; to confuse matters further, it is often called σ in the physics literature, and in this book it is τ). Now for each point on this plot, construct the plane perpendicular to the line from that point back to the origin and throw away everything beyond that plane. What is left after all those half spaces are discarded is W_Φ. See Figure 1.

In applications, $\Phi(\mathbf{n})$ is taken to be the surface free energy per unit area (a.k.a. surface tension) for a plane segment having oriented normal $\mathbf{n}$ separating one material from another. So, if one has a chunk S of one kind of stuff

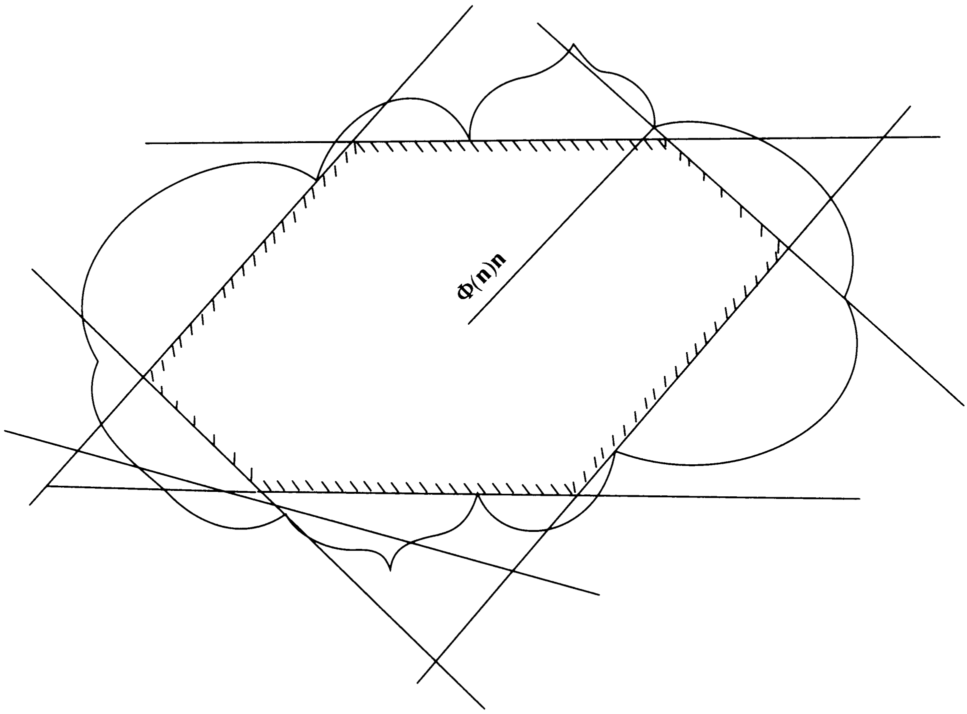

FIGURE 1. The Wulff construction for a fanciful (non-physical) Φ.

surrounded by a medium of something else and S has an exterior unit normal $\mathbf{n}_S(x)$ for almost all x in ∂S, then

$$\Phi(\partial S) = \int_{x \in \partial S} \Phi(\mathbf{n}_S(x))\, d\mathscr{H}^{n-1}x$$

is the total surface free energy of ∂S. If one or the other of the materials has an internal order, like crystals do, then different directions for surfaces are physically different, giving the $\mathbf{n}$ dependence (the crystal axes and the materials must be fixed in order to define Φ). If one rotates the crystal(s), one has to rotate W_Φ; and if one exchanges what is regarded as "behind" the surface with what is "in front", one reverses the orientation and thereby centrally inverts W_Φ. Φ also depends on the temperature.

If W_Φ is a compact set of positive volume, then among surfaces bounding regions of the same volume, the boundary of W_Φ is the unique minimizer for the integral of Φ, up to translations [W, B, H, T, F, DP, FM, BM]. If it is not a compact set of positive volume, then no minimum for the integral of Φ exists among compact regions of a given volume.[1] The Wulff shape is often called the

[1]The reader may wonder if we do not have to put some condition on Φ in order to be able to integrate it over, say, a rectifiable set; but in proving this result we can restrict ourselves without loss of generality to continuous Φ and even to the convex function Φ^c with the same Wulff shape as Φ (i.e., $\Phi^c(\mathbf{n}) = \sup\{x \cdot \mathbf{n} \colon x \in W_\Phi\}$ and $\Phi^c(\mathbf{p}) = |\mathbf{p}|\Phi^c(\mathbf{p}/|\mathbf{p}|)$). Also, if W_Φ is compact with positive volume, then one can pick an interior point a of the set and replace Φ by its translation Φ_a defined by $\Phi_a(\mathbf{n}) = \Phi(\mathbf{n}) - a \cdot \mathbf{n}$; the Wulff shape of Φ_a is then the translation by a of W_Φ, and Φ_a is positive on all unit vectors. According to the Stokes Theorem, the integral of Φ_a over

equilibrium crystal shape because of this property of having the least surface energy for the volume enclosed. In fact, Φ is often experimentally determined by looking at shapes under the microscope and working backward, although as the authors note, "It is actually not easy to bring a crystal into equilibrium with the surrounding vapour or melt...relaxation times are very long even for tiny crystals."

The Wulff construction has connections to analysis, geometry, statistical mechanics, physics, materials science, and crystallography and has been, directly or peripherally, a subject of research in these areas for a hundred years. I have been told that it was devised independently by G. Wulff, P. Curie, and J. W. Gibbs and is sometimes called the Gibbs-Curie-Wulff construction.

The contribution of this book to the long literature is that it provides the proof in the context of the Ising ferromagnet model of statistical mechanics that the expected shape of a "droplet" of fixed total spin in a finite square in $\mathbf{R}^2$ is essentially the Wulff shape W_Φ for the surface tension function Φ given naturally by the model. In the authors' words, their objective is thus to "justify the Wulff construction directly from a microscopic theory." In the process it extends the Bonnesen inequality in $\mathbf{R}^2$, a result independent of statistical physics. (See [G] for another recent extension and [P] for related work on the Wulff shape in the two-dimensional Ising model.) All these results use methods that are strictly two-dimensional.

What this book does

Let Z^2 be the two-dimensional integer lattice, and let $T_N = Z^2/NZ^2$; let $\Omega_N = \{-1, 1\}^{T_N}$ be the set of all configurations. That is, take a two-dimensional $N \times N$ square lattice with periodic boundary conditions for everything, and mark each site with a plus sign or a minus sign. Those sites with a plus constitute one phase, and those with a minus another phase. For fixed N, require a fixed excess of positive sites R_N. Limits for $N \to \infty$ are taken with sequences such that $R_N/N^2 \to \rho$ for some prescribed ratio ρ. See Figure 2.

An energy of interaction between pairs of sites s_1, s_2 is given by a function $U(s_1 - s_2)$. The distance between sites is given by the norm on T_N introduced by $|t| = |t^1| + |t^2|$ on Z^2. The Ising model has $U(t) = 1$ if $|t| = 1$ and 0 otherwise (i.e., only nearest neighbor interactions). In their introduction, the authors formulate results for the case of general finite-range ferromagnetic (nonnegative, even) potentials, but for proofs in the remaining chapters they consider only the case of the Ising model. They sketch the extension to general ferromagnetic models at the end of the final chapter.

The authors study the thermodynamic limit of canonical ensembles that are simultaneously rescaled to a unit volume. They get a limiting measure, with clear-cut regions of opposite phases separated by the boundary of the Wulff shape. Alternatively, they represent the microscopic configuration in terms of contours and prove that with probability nearly one there is only one long contour in a configuration and that this contour does not deviate too much from the Wulff contour and separates opposite phases. In the regions distant enough

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any surface without boundary is the same as the integral of Φ, and for surfaces with the same boundary the difference in integrals depends only on the boundary. Thus, if we want to, we can also assume that Φ is positive on unit vectors. Φ^c is the support function of W_{Φ^c} [R], and Φ_{Φ^c} is the unit ball for the dual norm $(\Phi^c)^*$.

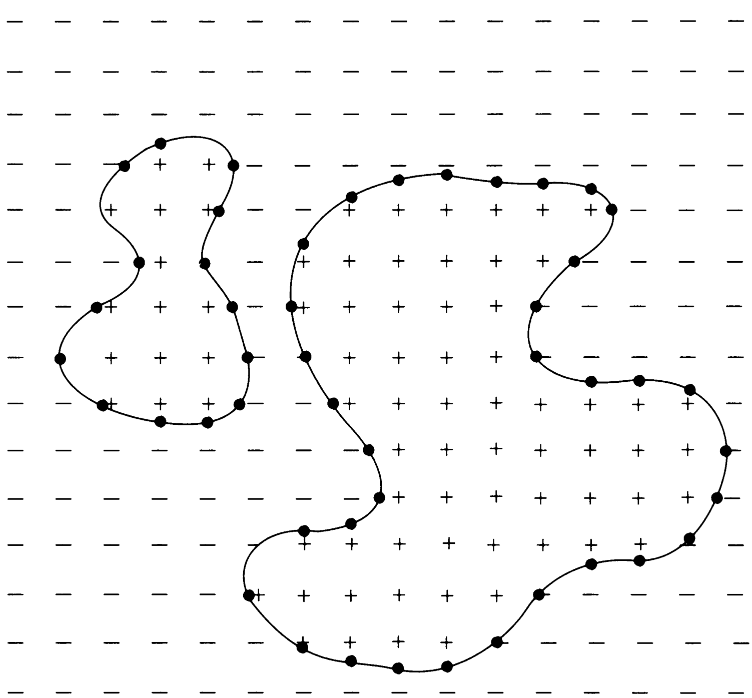

FIGURE 2. A configuration and its contours (adapted from Dobrushin et al., Figure 1.4).

from the contour, they show local variables behave as if in pure phases. (The authors give very precise but lengthy formulations of these statements.)

The authors throughout try to find natural general formulations and to exposit the basics (including well-known results on Ising models) so that the book is quite self-contained. The nature of the approach is indicated by the following brief summary of the chapters. Chapter 1: introduction, precise statements of two theorems, plan of the proof; Chapter 2: a generalization of the Bonnesen inequality; Chapter 3: accurate upper bounds on probabilities of large deviations of the sum of spins; assorted limit theorems; Chapter 4: results about surface tension such as the stability of the computed surface tension with respect to a change of the form of the volume (it turns out that the surface tension is an elliptic parametric integrand); Chapter 5: results about large contours; Chapter 6: proof of the main results.

SOME OTHER CURRENT MATHEMATICS RELATED TO THE WULFF SHAPE

Recent continuum proofs of the Wulff construction [F, DP, FM, BM] show that the subject is as rich as the usual isoperimetric inequality. Particularly noteworthy is the Brothers and Morgan proof, which manages to reduce the proof to the arithmetic-geometric mean inequality by a clever choice of map from an arbitrary region of the correct volume to W_Φ. Wulff shapes also have been shown to arise from jamority voter models [GG].

The Wulff construction ties in to several different areas of mathematics, as is evident from the history of its proofs and the range of mathematicians who have been involved with it. One area is minimal surfaces and related geometric analysis shading into applications to materials science—one can study surfaces that are stationary for surface energy rather than for surface area and surface-energy-driven motion problems (such as analogs of motion by mean curvature, models for dendritic crystal growth). A possible entry into this literature is the text [M] and the survey article [TCH]. The mixed-volume, Minkowski-metric approach is represented by the work of Gage (e.g., [G]) on curves in the plane; in $\mathbf{R}^2$, Φ gives rise to a Minkowski (Finsler) metric by setting F on a unit tangent direction equal to Φ on the corresponding exterior normal direction. Also, macroscopic limits of stochastic Ising feromagnetic models in R^n with long range interactions and Glauber dynamics were studied recently in [KS], which showed in the limit the development of a sharp interface moving by mean curvature times an explicitly determined constant (surface free energy becomes isotropic in this model).

References

[BM] J. Brothers and F. Morgan, *The isoperimetric theorem for general integrands*, Michigan Math. J. (to appear).

[B] H. Busemann, *The isoperimetric problem for Minkowski area*, Amer. J. Math **71** (1949), 743–762.

[DP] B. Dacorogna and C. E. Pfister, *Wulff theorem and best constant in Sobolev inequality*, J. Math. Pures Appl. (9) **71** (1992), 97–118.

[D] A. Dinghas, *Über einen Geometrischen Satz von Wulff fur die Gleichgewichtsform von Kristallen*, Z. Kristall **105** (1944), 304–314.

[F] I. Fonseca, *The Wulff theorem revisited*, Proc. Roy. Soc. London Ser. A **432** (1991), 125–145.

[FM] I. Fonseca and S. Muller, *A uniqueness proof for the Wulff problem*, Proc. Edinburgh Math. Soc. **119A** (1991), 125–136.

[G] M. Gage, *Evolving plane curves by curvature in relative geometries*, Duke Math. J. **72** (1993), 441–466.

[GG] J. Gravner and D. Griffeath, *Threshold growth dynamics*, Trans. Amer. Math. Soc. **340** (1993), 837–870.

[H] C. Herring, *The use of classical macroscopic concepts in surface energy problems*, Structure and Properties of Solid Surfaces (R. Gomer, ed.), Univ. of Chicago Press, Chicago, 1952, pp. 5–73; *Some theorems on the free energy of crystal surfaces*, Phys. Rev. **28** (1951), 87–93.

[KS] M. Katsoulakis and P. E. Souganidis, *Interacting particle systems and generalized evolution of fronts*, Arch. Rational Mech. Anal. (in preparation).

[M] F. Morgan, *Geometric measure theory. A beginner's guide*, Academic Press, New York, 1988.

[O] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.

[P] C. E. Pfister, *Long deviations and phase separation in the two-dimensional Ising model*, Helv. Phys. Acta **64** (1991), 953–1054.

[R] R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press, Princeton, NJ, 1970.

[T] J. E. Taylor, *Existence and structure of solutions to a class of nonelliptic variational problems*, Sympos. Math., vol. 14, Academic Press, London, 1974, pp. 499–508; *Unique structure of solutions to a class of nonelliptic variational problems*, Proc. Sympos. Pure Math., vol. XXVII, Amer. Math. Soc., Providence, RI, 1974, pp. 481–489.

[TCH] J. E. Taylor, J. W. Cahn, and C. A. Handwerker, *Geometric models of crystal growth*, Acta Met. Mat. **40** (1992), 1443–1474.

[W] G. Wulff, *Zur frage der Geschwindigkeit des Wachstums und der Auflosung der Krystalflachen*, Z. Krist. **34** (1901), 449.

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Oriented matroids, by A. Björner, M. Las Vergnas, B. Sturnfels, N. White, and G. Ziegler. Cambridge University Press, London and New York, 1993, xii + 516 pp., $89.95. ISBN 0-521-41836-4

In 1935 H. Whitney introduced the concept of a matroid, which unifies many configurations studied in pure and applied mathematics—in particular, in algebra, geometry, combinatorics, and optimization theory. Ordinary matroids may be viewed as an abstraction of finite geometric configurations which are embedded into some vector space over a field. In 1978 oriented matroids were introduced by R. G. Bland, M. Las Vergans, J. Folkman, and J. Lawrence as combinatorial abstractions of finite geometric configurations in vector spaces over some ordered field. Since oriented matroids, as well as ordinary matroids, are important in many areas—as in algebraic and computational geometry combinatorics, topology, operations research, and chemistry—researchers in various fields were led to questions concerning oriented matroids. The purpose of the present book is to summarize the theory of oriented matroids developed thus far. Technically, the book is organized as follows:

Chapters I and II serve to motivate the definition of oriented matroids by means of connections to several branches of mathematics and natural sciences. The diverse mathematical theories all lead to cryptomorphic axiom systems for oriented matroids; the equivalence of these definitions is proved in Chapter III. It should be remarked that the proofs are not simple.

Chapters IV and V are devoted to topological representability of oriented matroids. The main results of these two chapters is the topological representation theorem which is already proved in the basic paper by J. Folkman and J. Lawrence concerning oriented matroids. It states roughly that the loop-free oriented matroids correspond to arrangements of generalized hyperplanes which are obtained from affine hyperplanes by certain topological deformations.

In Chapter VI arrangements of pseudolines are studied, and it is shown that they correspond to reorientation classes of simple orientable matroids of rank 3. Many examples are presented which, on the one hand, are not trivial but which, on the other hand, are simple to illustrate graphically. Moreover, some connections between oriented matroids and Grünbaum's exposition of pseudoline arrangements are described.