BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 2, October 1994

0273-0979/94 $1.00 + $.25 per page

Billiards. A genetic introduction to the dynamics of systems with impacts, by V. V. Kozlov and D. V. Treshchev. Amer. Math. Soc., Providence, RI, 1991, vii+171 pp. ISBN 0-8218-4550-0

The book has a captivating title: *Billiards*. What are billiards? The authors give their definition in the extension of the title; they are dynamical systems with impacts, i.e., with infinitely short interactions. The interacting bodies do not change their positions in the impact, but their velocities are changed instantaneously.

Mathematically speaking such dynamical systems are concatenations of flows and diffeomorphisms. We have a vector field V in the phase space $M \subset \mathbb{R}^N$. Most of the boundary ∂M splits naturally into two parts: $\partial^- M$ where V points outside of M and $\partial^+ M$ where V points inside. The description of the dynamical system is completed by a (local) diffeomorphism $\Phi: \partial^- M \to \partial^+ M$. The phase point follows the vector field V until it leaves the phase space M, at which time it is returned into the phase space by the diffeomorphism Φ. In [W1] such concatenations were given an ad hoc name of flows with collisions. One can now redo the Dynamical Systems Theory for such concatenations, but the interest of this project is not at all clear. The Dynamical Systems Theory is more about special classes of systems and typical (characteristic) phenomena.

The introduction of the book contains a variety of examples of mechanical systems with impacts. We also find there a fair amount of history of the subject. This is a very well-written chapter which will attract a wide audience.

Elastic collisions can be understood as a natural extension of the principle of least action. This variational principle is used as the basis of the dynamics of systems with ideal holonomic constraints, i.e., when the positions of the bodies are restricted to a submanifold in the configuration space. Another way to introduce holonomic constraints is to consider a strong potential force field pushing the system toward the submanifold representing the constraint. Taking the strength of the field ("the modulus of elasticity") to infinity, we obtain the usual Lagrange equations. In Chapter 1 the authors look at systems with impacts as mechanical systems with one-sided constraints and derive the laws of elastic and inelastic impacts from the above limiting procedure in the full equations of motions of the "freed" system. They call it the genetic method.

Chapter 2 is devoted to the periodic trajectories in the Birkhoff's billiard, i.e., the system of a point particle moving in a (convex) domain in $\mathbb{R}^2$ and colliding elastically with the boundary. If we consider the map "from collision to collision", we obtain an area-preserving twist map. It turns out that the length functional of the Birkhoff's billiard has a counterpart for any area-preserving twist map.

The authors give a detailed proof of the existence of a multitude of periodic orbits, a result going back to Birkhoff. This subject has an interesting history. Birkhoff gave two proofs of the existence of these periodic orbits. One was based on the Poincaré's Last Geometric Theorem, and it was readily applicable to area-preserving twist maps. The other was based on the variational argument, and Birkhoff acknowledged that he did not know how to generalize it to twist maps. Birkhoff's account of the variational method ([Bir], page 362) is sketchy (probably because of its perceived limited scope), and it is not clear to us whether he considered the space of inscribed polygons with the vertices ordered as in a rotation or with an arbitrary ordering. The authors consider for every number of collisions and every number of rotations the large space of configurations without regard for the ordering of the collision points. (It is a nice exercise to show that the maximal polygon in each of these classes has, by necessity, ordered vertices.) The monotonicity of the Birkhoff orbits is critically important for the structure of their limits, invariant curves, or Cantor sets. In the authors' approach the ordering (monotonicity) of the orbit is lost, but it prepares us for the generalization to the multidimensional case (briefly described in the chapter), where we lose the monotonicity anyway.

Contrary to Birkhoff's emphasis on the nonvariational methods, the variational approach proved very fruitful. It was developed by Mather in his theory of Cantori for measure-preserving twist maps. Aubry obtained the same results independently starting from a model in solid state physics. There is an excellent survey of the Aubry-Mather theory by Bangert [B] (where the early papers of Hedlund are also brought to light).

The rest of the chapter is devoted to the connections between the multipliers of the periodic orbit and the character of the respective critical point of the length functional. The connections between the dynamic and geometric properties of the periodic orbits are further expounded in Appendix II. The criteria of stability of a two-link billiard orbit are derived, and this is the first of the three derivations in the book. (Let us warn the readers that, contrary to the remark made after Proposition 4 on page 71, the proposition as formulated does not hold in the general case of arbitrary curvatures.)

In Chapter 3 the authors discuss Lyapunov's theory of stability and Hill's method. The genetic method comes back here. It allows the derivation of stability criteria by replacing the limit impact with prelimit differential equations. The benefits of such an approach are not so obvious. Our experience is that straightforward calculation of the monodromy matrix, if possible, is not more cumbersome, [W2].

Going beyond periodic orbits of a dynamical system, we enter the vast territory of global behavior, be it integrable or nonintegrable.

Chapter 4 is devoted to integrable billiards. It starts with the discussion of the billiards in an ellipse and their connection with the geodesic flow on the triaxial ellipsoid. The authors' approach is the following. The geodesic flow is integrable, and the billiard system is its limit as one of the axes goes to zero. One can then derive information about the billiard system from the formulas for the geodesic flow. In particular, the action-angle variables for the billiard are found and the frequencies of the quasiperiodic motions calculated. This is an analytic approach which allows a straightforward generalization to higher dimensions. We would like to describe briefly here the work of Kołodziej [K], who found several purely geometric ways to determine the frequencies. One of them is related to Poncelet's Theorem. The twist map corresponding to the elliptic billiard has a continuous family of invariant curves. The invariant measure (area) for the twist map induces invariant measures (arc length) on the invariant curves. A mapping of a circle which preserves an arc length is a rotation. Hence, if we know the invariant arc length, we can immediately calculate the rotation numbers (frequency ratios). Geometrically the invariant curves (not all) can be represented by ellipses confocal with the boundary. For any such ellipse, let us consider the mapping of the boundary onto itself, which takes one endpoint of the orbit segment, tangent to the interior ellipse, into another. In view of the presence of the invariant arc length for this mapping, we get Poncelet's Theorem for this special pair of conics (for a rotation either every orbit is periodic or every orbit is dense, which is exactly what is claimed in the Poncelet Theorem). Kołodziej made two observations. On one hand, for a pair of (in general nonconcentric) circles, one inside the other, the invariant arc length for the mapping has the obvious density (the inverse of the distance to the point of tangency with the inner circle). On the other hand, any two ellipses (one of which is inside the other and not necessarily confocal) can be mapped onto two circles by a projective transformation of the plane (Kołodziej gives a simple geometric construction of the projective map). The claim of the Poncelet Theorem remains valid under the projective transformation, so it is sufficient to establish it for the pair of circles. The invariant arc length for the pair of circles will be transported to the invariant arc length for a general pair of ellipses.

In Chapter 5 nonintegrability is the subject. The brief discussion of ergodic billiards in Chapter 5 is outdated. The theory of polygonal billiards by Kerckhoff, Masur, and Smillie can be found in [KMS]. New results on (locally) convex billiards with hyperbolic behavior in all of the phase space can be found in [W3] and [D]. The surveys [ChS, W4, and LW] present some of the new approaches to systems with elastic collisions which are hyperbolic in all of the phase space. It is interesting to note that conservative systems with good stochastic properties in the entire phase space are exceptional and the known examples are mostly those of systems with elastic collisions. One explanation is that the discontinuities introduced by the multiple or "grazing" collisions may free us immediately from the consequences of the KAM theory, which excludes even the ergodicity of the system.

Overall, it is a stimulating and enjoyable book. However, it is not a monograph on the subject; the authors take us only where they want to go. The focus is on the authors' contributions and, more generally, on results of the Russian school. We get a book which should be read and enjoyed but not necessarily relied on as a comprehensive source.

It is our point of view that there is no "theory of billiards". Billiards are studied by the same methods that can be applied to much larger classes of dynamical systems. At the same time it is striking how influential the study of billiards actually was. The following two cases stand out.

Birkhoff's work on billiards in a plane convex domain and Hedlund's work on geodesics on tori were the precursors of the Aubry-Mather theory of Cantori.

Sinai's work on the Boltzman gas of hard balls, or on what is not known as Sinai's billiards, had a crucial influence in the development of the Pesin theory, which translates the infinitesimal picture furnished by the Oseledet's Multiplicative Ergodic Theorem into the information about the dynamics of nonlinear systems. In the future one may expect a unified theory of piecewise differentiable dynamical systems with hyperbolic behavior, covering both conservative and dissipative cases.

Finally, let us draw the attention of the readers to a recent very interesting survey of billiards by Tabachnikov [T].

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AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 2, October 1994

0273-0979/94 $1.00 + $.25 per page

Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations, by J. Palis and F. Takens, Cambridge Studies in Advanced Mathematics, vol. 35, Cambridge University Press, Cambridge, 1993, x+234 pp., $54.95. ISBN 0-521-39064-8

The fact that fairly simple dynamical systems can exhibit remarkably rich complexity in their qualitative behavior may have been known before Poincaré, but it was he who first discussed one of the archetypal examples in an essay on the stability of the solar system written around 1890.

This kind of behavior (the name "chaos" is currently very much in vogue) occurs in a very simple model problem illustrated in Figure 1. This figure describes the behavior of a differentiable, invertible self-map f of the plane R^2.