Finally, let us draw the attention of the readers to a recent very interesting survey of billiards by Tabachnikov [T].

REFERENCES

[B] V. Bangert, *Mather sets for twist maps and geodesics on tori*, Dynamics Reported **1** (1988), 1–56.

[Bir] G. D. Birkhoff, *Collected mathematical papers, Vol.* 2, Dover, New York, 1968.

[CS] N. I. Chernov and Ya. G. Sinai, *Ergodic properties of some systems of 2-dimensional discs and 3-dimensional spheres*, Russian Math. Surveys **42** (1987), 181–207.

[D] V. Donnay, *Using integrability to produce chaos: Billiards with positive entropy*, Comm. Math. Phys. **141** (1991), 225–257.

[K] R. Kołodziej, *The rotation number of some transformation related to billiards in an ellipse*, Studia Math. **81** (1985), 293–302.

[KMS] S. Kerckhoff, H. Masur, and J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math. (2) **124** (1986), 293–311.

[LW] C. Liverani and M. P. Wojtkowski, *Ergodicity in Hamiltonian systems*, Dynamics Reported (to appear).

[T] S. Tabachnikov, *Billiards*, preprint, Univ. of Arkansas, 1994.

[W1] M. P. Wojtkowski, *A system of one dimensional balls with gravity*, Comm. Math. Phys. **126** (1990), 507–533.

[W2] ———, *The system of two spinning disks in the torus*, Phys. D. **71** (1994), 430–439.

[W3] ———, *Principles for the design of billiards with nonvanishing Lyapunov exponents*, Comm. Math. Phys. **105** (1986), 391–414.

[W4] ———, *Systems of classical interacting particles with nonvanishing Lyapunov exponents*, Lyapunov Exponents, Proceedings, Oberwolfach 1990 (L. Arnold, H. Crauel, and J.-P. Eckmann, eds.), Lecture Notes in Math., vol. 1486, Springer-Verlag, New York, 1991, pp. 243–262.

MACIEJ P. WOJTKOWSKI
UNIVERSITY OF ARIZONA
E-mail address: maciejw@math.arizona.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 2, October 1994

0273-0979/94 $1.00 + $.25 per page

Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations, by J. Palis and F. Takens, Cambridge Studies in Advanced Mathematics, vol. 35, Cambridge University Press, Cambridge, 1993, x+234 pp., $54.95. ISBN 0-521-39064-8

The fact that fairly simple dynamical systems can exhibit remarkably rich complexity in their qualitative behavior may have been known before Poincaré, but it was he who first discussed one of the archetypal examples in an essay on the stability of the solar system written around 1890.

This kind of behavior (the name "chaos" is currently very much in vogue) occurs in a very simple model problem illustrated in Figure 1. This figure describes the behavior of a differentiable, invertible self-map f of the plane R^2.

The point p is fixed under f, and in a small neighborhood of p coordinates can be chosen so that $f(x, y) = (ax, by)$ where $a > 1$ and $0 < b < 1$. In the neighborhood where this formula is valid it is clear that the x- and y-axes are invariant and that f expands the x-axis while contracting the y-axis. It follows that if we "iterate" f on an interval I in the x-axis containing p, we will obtain an invariant curve

$$W^{u}(p) = \bigcup_{n=0}^{\infty} f^{n}(I)$$

called the *unstable manifold* of p. The *stable manifold* $W^{s}(p)$ is defined similarly using f^{-1} and an interval in the y-axis.

As these curves leave the neighborhood of p where the linear representation of f is valid, they may cease to be straight lines and can in fact bend and cross at a point q as shown in Figure 1. It is easy to see that

$$\lim_{n\to\infty} f^{-n}(q) = p \quad \text{and} \quad \lim_{n\to\infty} f^{n}(q) = p.$$

The point q and other points with this property are said to be *homoclinic points* associated to the fixed point p. They are called *transversal* homoclinic points because they are the points of intersection where the stable and unstable manifolds cross as opposed to being tangent. There are some remarkable consequences of this simple situation. Since q is a point where $W^{u}(p)$ and $W^{s}(p)$ cross and these curves are left invariant by f, *every point* $f^{n}(q)$ of the orbit of q must be a point where these curves cross each other. At the same time the curves $W^{u}(p)$ and $W^{s}(p)$ have no self-intersections. One can show that if we parameterize $W^{u}(p)$, say, by the real numbers, then every point in $W^{u}(p)$ is a limit of a sequence of points in $W^{u}(p)$ which tend to infinity in the parameterization. It is easy to see this for the point p, since $\{f^{n}(q)\}$ is such a sequence. It is an interesting exercise to sketch more of $W^{u}(p)$ and $W^{s}(p)$ than is shown in Figure 1.

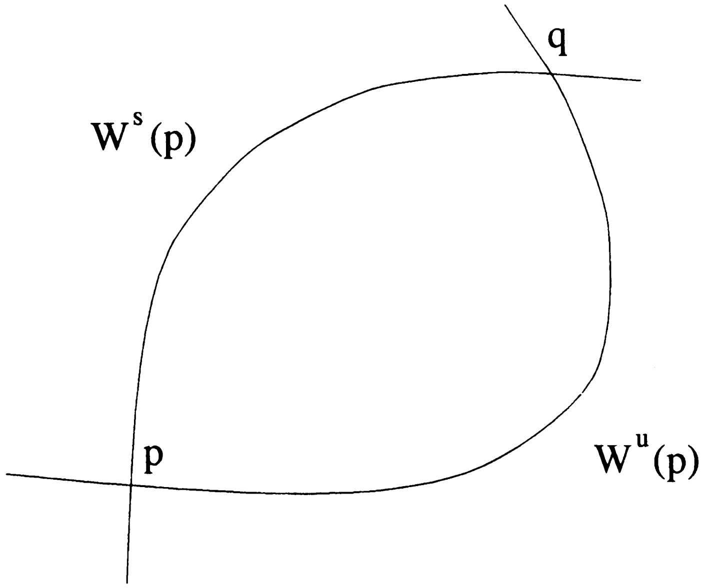

FIGURE 1

The richness of this example is attested to by the fact that much of the book of Palis and Takens is centered around it. More precisely the book focuses on the dynamics of maps in a smooth one-parameter family f_μ with parameter $\mu \in [0, 1]$ which starts at $\mu = 0$ with very simple behavior where $W^u(p)$ and $W^s(p)$ intersect only at p and ends with the example f above when $\mu = 1$.

An early chapter of the book is devoted to an important discovery of Smale that often much of the dynamical complexity inherent in transversal homoclinic points can be described very nicely with what is called *symbolic dynamics*. As an example, consider the space Σ of bi-infinite sequences of the "symbols" 0 and 1, i.e., all sequences of zeros and ones indexed by the integers. This space is a countable Cartesian product of the discrete space consisting of the two points 0 and 1. Hence, it is easily seen to be compact and metrizable. A natural homeomorphism of Σ is the shift map $\sigma: \Sigma \to \Sigma$ which shifts all the symbols in a sequence one place, say, to the left. (So the nth element of a sequence in Σ will be the $(n-1)$st element of its image under σ.)

Much of the dynamics of σ is quite easy to understand. For example, there are two fixed points: the sequence of all zeros and the sequence of all ones. Periodic points correspond to periodic sequences. There are points with dense orbits under σ (the reader may want to attempt the not-too-difficult exercise of exhibiting one). Associated to the fixed point whose sequence consists of all zeros are points with the same limit properties as p and q described above. These are precisely with those sequences $\{a_n\}$ for which there is an integer N with $a_n = 0$ for all $n > N$ or $n < -N$, and they are naturally enough called homoclinic points.

The remarkable discovery of Smale was that for examples like f described above (and, in fact, much more generally) there are neighborhoods U and V of p and q respectively and an iterate f^n of f with the property that dynamics of those points whose orbits stay in these neighborhoods is identical to the dynamics of the shift map σ. More precisely, if

$$\Lambda = \bigcap_{k=-\infty}^{\infty} f^{kn}(U \cup V),$$

then there is a homeomorphism $h: \Sigma \to \Lambda$ such that $h \circ \sigma = f^n \circ h$. Such an h is called a topological conjugacy, since $\sigma = h^{-1} \circ f^n \circ h$. It is easy to see that h carries dynamically significant sets (fixed points, periodic points, dense orbits, homoclinic orbits, etc.) from one system to sets of the corresponding type for the other system.

This material is available in many graduate level texts on dynamics, but two other important topics treated by Palis and Takens are not readily available outside their original sources. The first of these is a result of Newhouse dealing with the attracting periodic orbits or *sinks*. A sink is a periodic point x with the property that for every y in some neighborhood of x it is the case that $\lim_{n\to\infty} f^{np}(y) = x$. An optimistic view, commonly held, prior to 1970 was that only artificially constructed examples could have infinitely many sinks and, just as a smooth real-valued function on a compact manifold typically has only finitely many critical points, perhaps diffeomorphisms would typically have only finitely many sinks. (This analogy is not as farfetched as it may initially seem, because a large class of diffeomorphisms can be constructed by flowing along the gradient lines of a real-valued function. For these systems the sinks are precisely the maxima of the function, which are, of course, critical points.)

A series of results of Newhouse showed that this optimism was not well founded. The key example is close to the one described above and illustrated in Figure 1 but with a crucial difference. In this figure the point q is a *transversal* homoclinic point, i.e., a point where the curves $W^{u}(p)$ and $W^{s}(p)$ cross and are not tangent. Newhouse considered a point where they intersect and are tangent, i.e., a *tangential homoclinic point.* He showed that if, in the example above, the expanding and contracting factors a and b have product less than 1 and q is a tangential homoclinic point, then close to f there are diffeomorphisms with infinitely many sinks. More importantly, there is a whole open set (in the space of diffeomorphisms with an appropriate topology) with the property that typically (in the sense of Baire category) diffeomorphisms in this set have infinitely many sinks. No nice result like the one for critical points of real-valued functions is possible.

The second important topic treated by Palis and Takens draws heavily from their own research in dynamics. It is the very important question of how this complex dynamic behavior can be created. More precisely, suppose that we consider not just the one function f illustrated in Figure 1 but also a smooth one-parameter family, f_{μ}, with parameter $\mu \in [0, 1]$ which starts at $\mu = 0$ with very simple behavior where $W^{u}(p)$ and $W^{s}(p)$ intersect only at p and ends with $f_1 = f$.

At how many parameter values must there be a *bifurcation*, i.e., a qualitative change in the dynamics of the function on the set Λ which is the closure of homoclinic points? The answer to this question, too, is intimately tied up with homoclinic tangencies (which must occur at some parameter values). There is an interesting new ingredient, however, the Hausdorff dimension of the set Λ. It turns out that if one investigates the set of parameters B where a bifurcation occurs near a particular parameter value μ_0 where f_{μ} has a homoclinic tangency when $\mu = \mu_0$ and satisfies some reasonable technical assumptions, the relative density of B near μ_0 is dependent on this Hausdorff dimension. More precisely, if this Hausdorff dimension is less than 1, then

$$\lim_{\mu \to \mu_0} \frac{m(B \cap [\mu_0, \mu])}{\mu - \mu_0} = 0;$$

while if the Hausdorff dimension is greater than 1,

$$\lim_{\mu \to \mu_0} \frac{m(B \cap [\mu_0, \mu])}{\mu - \mu_0} > 0,$$

where $m(\ )$ denotes Lebesgue measure.

This is a fascinating result showing a surprising relation between the relative density of the bifurcation set and the dimension of the set on which the dynamics is occurring for one particular parameter.

The book by Palis and Takens is an interesting monograph on this collection of ideas. It begins with material often covered in graduate texts but quickly moves to the exposition of ideas only available in the original sources. It would be quite suitable for an advanced graduate level course in dynamics and bifurcation theory.

JOHN FRANKS
NORTHWESTERN UNIVERSITY
E-mail address: john@math.nwu.edu

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, Number 2, October 1994

0273-0979/94 $1.00 + $.25 per page

Dirac structures and integrability of nonlinear evolution equations, by Irene Dorfman. Nonlinear Science: Theory and Applications, Wiley & Sons, New York, 1993, vii+176 pp., $75.00. ISBN 0-471-93893-9

In the beginning (1834), Hamilton created the Hamiltonian system and thereby quantified light. (Well, actually, Hamilton was anticipated by Poisson, Lagrange, and others (see [1, p. 264]), but let us not quibble.) Hamilton's equations might have remained a somewhat esoteric curiosity, of interest to specialists in optics and mechanics, were it not for their sudden and unexpected starring role in Schrödinger's wave mechanics theory of quantization. (In hindsight, it is remarkable how close Hamilton came to quantum mechanics, lacking only the physical motivation for introducing a wave theory of matter!) Initially, Hamiltonian systems were always written in terms of canonical coordinates, the p's and q's of classical mechanics; indeed, for basic quantization, this reliance on a particular coordinate system was essential. Moreover, an old theorem of Darboux (originally stated for one-forms) implies that one can always find canonical coordinates, so (at least in the finite-dimensional framework) there was initially no reason to dispense with these canonical coordinates.

In recent years, though, coordinate-free approaches to Hamiltonian mechanics have finally come into their own. In part, this process was motivated by the discovery of mechanical systems (the simplest being Euler's equations for the rigid body) which do not naturally fall into the traditional framework (the classical Hamiltonian approach to the Euler equations being rather forced). A second important factor was the discovery of important infinite-dimensional Hamiltonian systems, particularly the equations of fluid mechanics and of soliton theory, for which a general Darboux theorem is not so apparent. In both cases, the introduction of Hamiltonian structures, both degenerate and of variable rank, necessitated a reassessment of the foundations of the subject. There arose two different, essentially dual approaches, each relying on a different object as the fundamental basis of Hamiltonian mechanics. The earlier approach is via the geometrical theory of symplectic mechanics, in which one introduces a closed, nondegenerate two-form ω—the symplectic form—which is a section of $\bigwedge^2 T^*M$, where M denotes the underlying phase space. The second, dual approach is to rely on the Poisson bracket as the primary object of interest. Geometrically, this amounts to the introduction of a nondegenerate bivector field Θ, which is a section of $\bigwedge^2 TM$. The closure condition of the symplectic two-form, which is equivalent to the all-important Jacobi identity for the associ-