moves to the exposition of ideas only available in the original sources. It would be quite suitable for an advanced graduate level course in dynamics and bifurcation theory.

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Dirac structures and integrability of nonlinear evolution equations, by Irene Dorfman. Nonlinear Science: Theory and Applications, Wiley & Sons, New York, 1993, vii+176 pp., $75.00. ISBN 0-471-93893-9

In the beginning (1834), Hamilton created the Hamiltonian system and thereby quantified light. (Well, actually, Hamilton was anticipated by Poisson, Lagrange, and others (see [1, p. 264]), but let us not quibble.) Hamilton's equations might have remained a somewhat esoteric curiosity, of interest to specialists in optics and mechanics, were it not for their sudden and unexpected starring role in Schrödinger's wave mechanics theory of quantization. (In hindsight, it is remarkable how close Hamilton came to quantum mechanics, lacking only the physical motivation for introducing a wave theory of matter!) Initially, Hamiltonian systems were always written in terms of canonical coordinates, the p's and q's of classical mechanics; indeed, for basic quantization, this reliance on a particular coordinate system was essential. Moreover, an old theorem of Darboux (originally stated for one-forms) implies that one can always find canonical coordinates, so (at least in the finite-dimensional framework) there was initially no reason to dispense with these canonical coordinates.

In recent years, though, coordinate-free approaches to Hamiltonian mechanics have finally come into their own. In part, this process was motivated by the discovery of mechanical systems (the simplest being Euler's equations for the rigid body) which do not naturally fall into the traditional framework (the classical Hamiltonian approach to the Euler equations being rather forced). A second important factor was the discovery of important infinite-dimensional Hamiltonian systems, particularly the equations of fluid mechanics and of soliton theory, for which a general Darboux theorem is not so apparent. In both cases, the introduction of Hamiltonian structures, both degenerate and of variable rank, necessitated a reassessment of the foundations of the subject. There arose two different, essentially dual approaches, each relying on a different object as the fundamental basis of Hamiltonian mechanics. The earlier approach is via the geometrical theory of symplectic mechanics, in which one introduces a closed, nondegenerate two-form ω—the symplectic form—which is a section of $\bigwedge^2 T^*M$, where M denotes the underlying phase space. The second, dual approach is to rely on the Poisson bracket as the primary object of interest. Geometrically, this amounts to the introduction of a nondegenerate bivector field Θ, which is a section of $\bigwedge^2 TM$. The closure condition of the symplectic two-form, which is equivalent to the all-important Jacobi identity for the associated Poisson bracket, can be dually formulated as the vanishing of the Schouten bracket $[\boldsymbol{\Theta}, \boldsymbol{\Theta}] = 0$. (The Schouten bracket is the unique natural extension of the Lie bracket between vector fields to multivector fields.)

In the applications to rigid body mechanics, fluid mechanics, and solitons, the introduction of degenerate and variable-rank Hamiltonian structures is required. In both approaches, one maintains the same basic structure but relaxes the nondegeneracy condition. Thus, to generalize the symplectic formulation, one relies on a "presymplectic" two-form, which is just a general closed two-form ω. In local coordinates, a (finite-dimensional) presymplectic Hamiltonian system takes the form

$$K(x)\frac{dx}{dt} = \nabla H(x),$$

in which $H(x)$ is the Hamiltonian function and $K(x)$ is the skew-symmetric structure matrix defining the presymplectic form $\omega = dx \wedge K dx$. Similarly, the Poisson approach uses a general Poisson bracket or, equivalently, bivector field with vanishing Schouten bracket. In local coordinates, a (finite-dimensional) Poisson system takes the form

$$\frac{dx}{dt} = J(x)\nabla H(x),$$

in which the skew-symmetric structure matrix $J(x)$ defines the Poisson bracket $\{F, G\} = \nabla F \cdot J\nabla G$ and associated bivector $\boldsymbol{\Theta} = \partial_x \wedge J\partial_x$. Of course, in the nondegenerate (symplectic) case, $J = K^{-1}$, and the two formulations are entirely equivalent.

The book under review is devoted to a further generalization, known as a Dirac structure, that serves to unify and extend both of these approaches. The basic definition begins by noting that the presymplectic form defines a linear map $\widehat{K}: TM \to T^*M$ from the tangent bundle to the cotangent bundle of the underlying phase space M, whose local coordinate formula is given by the structure matrix K. Vice-versa, a Poisson structure defines a linear map $\widehat{J}: T^*M \to TM$, whose local coordinate formula is given by the dual structure matrix J. In the latter case, the differential of the Hamiltonian function dH is mapped to its associated Hamiltonian vector field $\mathbf{v}_H = \widehat{J}(dH)$ whose flow defines Hamilton's equations. In both cases, the *graph* of the linear map forms an $m = \dim M$ dimensional linear subspace of the tensor product bundle $TM \otimes T^*M$. A Dirac structure, then, is determined by more general m-dimensional subspaces of $TM \otimes T^*M$. The skew-symmetry requires that the subspace be isotropic with respect to a natural bilinear form. The Jacobi identity or closure condition requires the vanishing of the "Nijenhuis torsion" of this subspace. The local coordinate form of a Dirac system then is

$$K(x)\frac{dx}{dt} = J(x)\nabla H(x),$$

in which both J and K may be degenerate.

Dirac structures were first introduced in the finite-dimensional context by T. Courant and A. Weinstein, although without significant physical motivation. In collaboration with I. M. Gel'fand, the author of the present book extended these constructions to infinite-dimensional systems, with particular emphasis on nonstandard soliton systems. Here the Hamiltonian system becomes a system of evolution equations, with the structure matrices $J(x)$ and $K(x)$ replaced by differential operators, which may depend on the field variables. An important example, which does not fit into either the presymplectic or Poisson framework, is the soliton equation

$$u_t = u_{xxx} - \frac{3u^2xx}{2u_x},$$

known as the singularity manifold or Krichever-Novikov equation.

Dorfman's book is a clearly written and comprehensive introduction to the theory of Dirac structures in the infinite-dimensional context, illustrated by a number of applications. It begins with a very abbreviated introduction to the basics of Hamiltonian systems and clearly assumes that the reader is already quite familiar with the material. The general theory is formulated rather abstractly, so the reader may at first be dismayed by the level of abstraction that the author has chosen to use. However, this is mitigated by the large variety of interesting, explicit examples chosen to illustrate the general constructions. These include the usual suspects from soliton theory, as well as several more exotic integrable systems.

In finite dimensions, Darboux's Theorem guarantees that (provided the rank is constant) canonical coordinates can always be introduced. For the infinite-dimensional Hamiltonian systems governed by evolution equations, the Hamiltonian structure is governed by a differential operator (or its inverse or, in the general Dirac case, a hybrid operator), and a general Darboux theorem is not known. (Weinstein's infinite-dimensional version of Darboux's Theorem [2] does not seem to be particularly applicable in this more formal context.) Thus, there remain many open questions concerning the general structure of Hamiltonian differential operators. As the book discusses, only low-order Hamiltonian operators have, so far, been completely classified; the general classification problem appears to be very difficult.

A large part of the book under review is devoted to the understanding of biHamiltonian structures and integrability for a wide variety of evolution equations. Magri [3] first discovered the striking result that if a system of differential equations can be written in Hamiltonian form in two different, compatible ways, then it admits an infinite hierarchy of mutually commuting symmetries and associated conservation laws in involution. Consequently, if enough of these are independent, then the system is completely integrable in the classical sense of Liouville. (The converse question of whether every completely integrable Hamiltonian system is biHamiltonian has been the subject of recent work of Brouzet [4] and Fernandes [5], who give obstructions for the existence of a biHamiltonian structure in the neighborhood of an invariant torus.) The biHamiltonian route to integrability was first recognized in the infinite-dimensional context of soliton equations, but, subsequently, its importance in the finite-dimensional context has been amply demonstrated. Indeed, the biHamiltonian system associated with the Lie-Poisson structure on simple Lie algebras resulting from the so-called R-matrix theory has been a prime motivation for Drinfel′d's profound theory of quantum groups [6]. The precise role of the compatibility conditions is unexplained, since all known examples of incompatible biHamiltonian systems are, in a certain sense, even more integrable than the compatible ones [7].

In summary, I found Dorfman's treatise clearly written and would warmly recommend it to the researcher or advanced student studying the geometrical approach to infinite-dimensional Hamiltonian systems and solitons.

Note: A month after I completed the text of this review, I was greatly saddened to learn that Irene Dorfman died from cancer. Her untimely death is a disheartening loss to mathematical physics.

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Contact geometry and linear differential equations, by V. Nazaikinskii, V. Shatalov, and B. Y. Sternin. Walter de Gruyter, Berlin, 1992, vii+216 pp., $69.00. ISBN 3-11-013381-4

During the first decades of this century, the investigations of asymptotic oscillatory solutions for the Schrödinger equation in quantum mechanics and for the wave equation in geometrical optics were mainly performed by using the W-K-B method and the method of characteristics. More precisely, the expected asymptotic solutions were of the form $e^{i\tau\varphi(x)}a(x)$ where φ is classically known as the phase function and a the amplitude function.

Since then, these methods have spread to the general theory of differential equations thanks to the research of many mathematicians mainly motivated by theoretical physics (see, for example, the works of Voros [8] and Sato, Kawai, and Kashiwara [7] and the references in the book under review).

The basic idea is to perform a "transformation" of the equations, which is also the idea of the classical Fourier transform (denoted by $\hat{}$): By defining $\hat{u}$