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NICOLA GAROFALO
PURDUE UNIVERSITY
E-mail address: garofalo@math.purdue.edu

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Nearrings—geneses and applications, by James R. Clay. Oxford University Press, London, 1992, x+469 pp., $95.00. ISBN 0-19-853398-5

A *nearring* (or near-ring) satisfies all axioms of an associative ring, except commutativity of addition and one of the two distributive laws. If the nearring N satisfies the left distributive law $a(b+c) = ab + ac$, then N is called a *left nearring.* A *right nearring* is, of course, a nearring satisfying the right distributive law. If the (left or right) nearring N satisfies the condition $On = nO = O$ for all $n \in N$, where O denotes the neutral element of the group $(N, +)$, then N is called *O-symmetric.* A nearring N with the property that $(N\backslash\{O\}, \cdot)$ is a group is called a *nearfield.* The additive group of a nearfield is always abelian. In this review, all nearrings under consideration will be left nearrings.

The first and fairly comprehensive treatise on nearrings by G. Pilz [P] appeared in 1977 and in revised form in 1983. The monograph by J. D. P. Meldrum, *Near-rings and their links with groups* [M], published in 1985, was intended as an introduction to the subject but also contains some deeper material on the group-theoretic aspect. The highly developed theory of nearfields is covered by Heinz Wähling's treatise *Theorie der Fastkörper* [W], which was published in 1987. How does the book under review fit in? Well, it is composed with the following explicit intentions:

(1) emphasis on applications;
(2) emphasis on foundations;
(3) minimal overlap with the existing books by Pilz, Meldrum, and Wähling.

The subtitle *geneses and applications* refers to (1) and (2). The readership that the author had in mind is advanced graduate students beginning their serious research. The author was very successful in achieving his goals. Despite the restrictions mentioned above, the book contains an impressive number of results and, in particular, recent results. We sketch the contents and choice of material.

Certainly, the core of the book is Chapter 2 on planar nearrings. (This chapter takes 172 of the total 469 pages.) The emphasis on this subject is very well founded: The theory of planar nearrings is one of the research areas of the author and, in fact, is to a considerable extent due to the author, and this branch of nearring theory provides most of the applications.

Let N be a left nearring. An equivalence relation $=_m$ on N is defined by

$$a =_m b \quad \text{iff} \quad ax = bx \text{ for all } x \in N.$$

In case of a nearfield, $=_m$ is equality. The nearring N is called *planar*, iff:

(i) $=_m$ has at least three equivalence classes;
(ii) for $a, b, c \in N$, with $a \neq_m b$, the equation $ax = bx + c$ has a unique solution for x in N.

Planarity of a nearfield F means that in the plane "over" F two nonparallel straight lines intersect in exactly one point. All planar nearrings are *O*-symmetric. There is a universal construction method for planar nearrings, due to Giovanni Ferrero. One starts from a (not necessarily commutative) group $(N, +)$ and a nontrivial group Φ of fixed-point free automorphisms of $(N, +)$ with the additional property: For all $\varphi \in \Phi\backslash\{1_N\}$, the mapping $-\varphi + 1_N$ is a surjection of N onto itself. In general, the pair (N, Φ) yields numerous planar nearrings with additive group $(N, +)$, and every planar nearring $(N, +, \cdot)$ may be constructed from such a "Ferrero pair" (N, Φ).

For a planar nearring N, one introduces the following sets of subsets of N:

$$\begin{aligned}
\mathscr{B} &:= \{Na + b \mid a, b \in N, a \neq 0\};\\
\mathscr{B}^* &:= \{N^*a + b \mid a, b \in N, a \neq 0\},\\
&\qquad \text{here } N^* := N\backslash\{a \in N \mid a =_m 0\};\\
\mathscr{B}^- &:= \{N\{a, -a\} + b \mid a, b \in N, a \neq 0\}.
\end{aligned}$$

These concepts are motivated by the following collection of examples: Let $(\mathbb{C}, +)$ be the additive group of complex numbers. Define three distinct multiplications on $\mathbb{C}$ by

$$\begin{aligned}
z *_1 a &:= \begin{cases} (\operatorname{Re} z)a & \text{if } \operatorname{Re} z \neq 0,\\ (\operatorname{Im} z)a & \text{if } \operatorname{Re} z = 0;\end{cases}\\
z *_2 a &:= |z|a;\\
z *_3 a &:= \begin{cases} \frac{z}{|z|}a & \text{if } z \neq 0,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Then $(\mathbb{C}, +, *_i)$ is a planar nearring, for $i = 1, 2, 3$. Now let $a, b \in \mathbb{C}$, $a \neq 0$. Then:

$\mathbb{C} *_1 a + b$ is the straight line through b and $a + b$;
$\mathbb{C}^* *_1 a + b$ is the same straight line without b;
$\mathbb{C} *_2 a + b$ is the ray from b through $a + b$;
$\mathbb{C} *_2 \{a, -a\} + b$ is again the straight line through b and $a + b$;
$\mathbb{C}^* *_3 a + b$ is the circle with center b and radius $|a|$.

Furthermore, $(\mathbb{C}, +, *_3)$ is a circular planar nearring. In general, a planar nearring $(N, +, \cdot)$ is called *circular*, iff

(i) every three distinct points (elements of N) belong to at most one "block" $N^*a + b$ from $\mathscr{B}^*$;
(ii) every two distinct points belong to at least two "blocks" from $\mathscr{B}^*$.

Finally, $(\mathbb{C}, +, *_2, *_3)$ is an example of a *double planar nearring*: the multiplications $*_2$ and $*_3$ are left distributive over each other.

An investigation of planar nearrings is to a considerable extent an investigation of the incidence structures

$$(N, \mathscr{B}, \in), (N, \mathscr{B}^*, \in), (N, \mathscr{B}^-, \in).$$

The class of circular nearrings is particularly important. If N is a finite planar nearring, then $(N, \mathscr{B}^*, \in)$ is always a balanced incomplete block design (BIBD). $(N, \mathscr{B}, \in)$ is a tactical configuration, and a necessary and sufficient condition is known for $(N, \mathscr{B}, \in)$ to be a BIBD as well.

These hints may suffice to indicate that the theory of planar nearrings, as presented in Chapter 2, is highly developed, with numerous links to and applications in combinatorics, coding theory, and cryptography. The author also included in this chapter the subject of sharply two-transitive groups and their corresponding neardomains (nearfields).

As for the rest of the book, a short list of contents may suffice. Chapter 1 is a brief and example-based introduction. Chapter 3 is devoted to nearrings linked with group or cogroup objects in a category. Chapter 4 describes some prominent types of nearrings and their ideals. Chapter 5 is devoted to the structure of groups of units of nearrings with identity. This is again a subject to which the author made substantial contributions. In the final chapter, Chapter 6, on "*Avant-garde* families of nearrings" the author presents "a small sampling of topics of current interest to researchers in nearring theory, but each of these topics shows promise of being a source for very interesting mathematics and a strong potential for applications" (quoted from p. 383).

As mentioned above, the book contains an impressive amount of material, mainly of current interest. Chapter 2 is a fairly complete presentation of the theory of planar nearrings and all its links to other fields.

The author took great pains to present the material clearly and to explain his points by many well-chosen examples. Other positive attributes are the "exploratory problems" and the numerous research problems indicated in the book.

Readers interested in nearring theory, combinatorics, finite geometry, and related topics will find the book very rewarding.

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G. BETSCH
UNIVERSITY OF TÜBINGEN
E-mail address: mmtbe01@mailserv.zdv.um-tuebingen.de

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Mathematical control theory: An introduction, by Jerzy Zabczyk. Birkhäuser, Boston, 1992, viii+260 pp., $69.50. ISBN 3-7643-3645-5

The tradition of book reviews for the *Bulletin* usually involves an exposition of the subject as well as a review of the book. In the present book Zabczyk discusses almost all of deterministic control theory, so a complete exposition is difficult. I will mention a few of the concepts and results from the part of the book on linear systems to try to give a flavor of the material.

A linear control system is modeled by an n-dimensional vector differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = a \tag{1}$$

describing the evolution of the n-dimensional state $x(t)$ of the system. The control $u(t)$ is an m-dimensional vector function of time which must be chosen to make the system behave in a desired manner. A and B are $n \times n$ and $n \times m$ matrices, and a is the initial state of the system.

In many situations it is desired to keep the system near zero. A square matrix A is called a stable matrix if all its eigenvalues have negative real parts. If A is a stable matrix, solutions of

$$\dot{x}(t) = Ax(t), \qquad x(0) = a \tag{2}$$

satisfy

$$|x(t)| \le Me^{-\omega t}|a| \tag{3}$$

for positive constants M and ω. Thus, in this case, all solutions of (2) decay exponentially to zero. The system (2) is said to be exponentially stable when (3) holds.

A control $u(t)$ is given by a linear feedback if it has the form

$$u(t) = Kx(t) \tag{4}$$

where K is an $m \times n$ matrix and $x(t)$ is the solution of (1) with this control. The problem of exponential stabilization of the linear system (1) is the problem of finding a matrix K so that $A + BK$ is a stable matrix. Thus when the linear