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Realism in mathematics, by Penelope Maddy. Oxford University Press, London, 1993, ix + 204 pp., $19.95. ISBN 0-19-824035-X

1. Realism in mathematics

> *Mathematics has always skirted dangerously close to the shores of metaphysics.*
>
> —S. G. Shanker
>
> *As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*
>
> —Einstein
>
> *Reality is just another model.*
>
> —Graffito in Evans Hall, Berkeley

As working mathematicians, we spend little time on philosophical issues. We would rather get down to work than waste time speculating on what it all means. Yet each of us must have wondered at some point—if only to reject the question as meaningless or too difficult—just what is mathematics about?

What are numbers? What is mathematical truth? In what sense does the number 3 exist, or π, or $2^{\aleph_0}$? If there were no life in the universe, would these numbers exist? Would the prime number theorem still be true? Is the Continuum Hypotheses definitely either true or false?

Penelope Maddy, professor of philosophy at University of California at Irvine, says in her introduction:

> Mathematicians, though privy to a wider range of mathematical truths than most of us, often incline to agree with unsullied common sense on the nature of those truths. They see themselves and their colleagues as investigators uncovering the properties of various fascinating districts of mathematical reality: number theorists study the integers, geometers study certain well-behaved spaces, group theorists study groups, set theorists sets, and so on.

Maddy refers to this common-sense attitude as "prephilosophical realism": Mathematics is a science of certain entities—numbers, sets, functions, and so forth—that really exist, just as physical science is the study of ordinary physical objects; and mathematical statements, being about reality, are either true or false.

The problem with this straightforward attitude, as Maddy points out, is that attempts to explain it run into embarrassing questions. What and where are these abstract objects? If, as Plato held, they are without location in space and time, then how can we know anything about them?

When faced with such formidable philosophical puzzles, many mathematicians retreat to a nihilistic formalism—"We are just playing meaningless games with empty symbols"—but none of us really believes that!

This "double-think" makes no difference to the mathematician, but it is not acceptable to the philosopher. Maddy's goal is to "develop and defend a version of the mathematician's pre-philosophical attitude." The purpose of this book is to justify a version of mathematical realism.

Belief in the objective existence of mathematical objects is called *Platonism.* But Maddy's position is far from Platonism in its strictest sense, namely, the view that mathematical entities are ideal forms completely outside physical space and time, eternal and unchanging, and that mathematical truths are a priori certain and necessary. On the contrary, she justifies the objective existence of mathematical concepts on the intimate relation between mathematics and science: Since physics is about real things and mathematics is indispensable for physics, then mathematics is also about real things. And we obtain knowledge of mathematical reality, such as sets, through our nervous systems.

2. Scientific realism

Entia non sunt multipicanda prater necessitatem.

—William of Occam

After discussing serious problems with the formalism, logicism and conventionalism, and dismissing intuitionism because "the job of the philosopher of mathematics is to describe and explain mathematics, not reform it," Maddy turns to mathematical realism. Her justification of it begins with W. V. O. Quine's *scientific realism* and his naturalist epistemology.

How do we justify our belief in the existence of esoteric objects and unobservable theoretical entities of science, like electrons and quarks, or the temperature of Pluto? For that matter, why do we believe in the objective existence of physical objects? Because such assumptions are part of science, our best way of understanding the physical world, and "being part of our best theory is the best justification a belief can have... what better justification would we have to believe in the most well-confirmed posits of our best scientific theory than the fact that they *are* the most well-confirmed posits of our best scientific theory?"

Even if we accept scientific realism, to thereby justify mathematical realism requires a link between physical science and mathematics. Here Maddy adopts an "indispensability" argument of Quine and Hilary Putnam: Because science is inconceivable without mathematics, "We are committed to the existence of mathematical objects because they are indispensable to our best theory of the world and we accept that theory."

Maddy points out that the Platonism arising from the Quine/Putnam indispensability argument is quite different from Plato's. While Plato considered mathematical knowledge to be a priori certain and necessary, the Quine/Putnam approach leads to no such conclusions: If mathematics is objective because it is embedded in scientific theory, it can hardly be considered a priori; and there is likewise little support for certainty or necessity.

But a more serious problem is that the Quine/Putnam account applies only to that part of mathematics used in science; it says nothing about most of "unapplied mathematics", which does not seem to be indispensable for physics. Quine accepts as objective "only that part of mathematics as is wanted for use in empirical science," along with things like "transfinite ramifications" which "come out of a simplificatory rounding out". But mathematicians, Maddy points out, "are not apt to think that the justification for their claims waits on the activities in the physics labs." We have our own ways of justifying our methods and conclusions, including proofs, intuitive evidence, plausibility arguments, and defenses in terms of consequences.

Maddy further objects that the mathematics justified by Quine/Putnam enters scientific theorizing only at fairly theoretical levels. We do not need physics to justify "$2 + 2 = 4$" or "the union of the set of even numbers with the set of odd numbers is the set of all numbers." In Charles Parsons's phrase, Quine/Putnamism "leaves unaccounted for precisely the *obviousness* of elementary mathematics." [1]

Here Maddy turns for help to Kurt Gödel, for whom the most elementary axioms of set theory "force themselves on us by being true." Gödel held that

> logic and mathematics (just as physics) are built up on axioms with a real content which cannot be "explained away".... The assumption of [sets] is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions.

But Gödel also objectified even nonintuitive mathematical truths, in analogy with physical facts about unobservable objects. Concerning the ways some new axiom might be justified, he claimed that even if it lacks intuitiveness, we might decide to accept it as true for the same kinds of reasons we accept a well-established physical theory: because of its power to prove verifiable consequences, obtain new results, and illuminate old ones.

Combining the ideas of Quine/Putnam and Gödel, Maddy arrives at *compromise Platonism*:

> From Quine/Putnam, this compromise takes the centrality of the indispensability arguments; from Gödel it takes the recognition of purely mathematical forms of evidence and the responsibility for explaining them. Thus it averts a major difficulty with Quine/Putnamism—its unfaithfulness to mathematical practice—and a major difficulty with Gödelism—its lack of a straightforward argument for the truth of mathematics.

There is still something missing, however. If "mathematical intuition" is to be a basis for an epistemology of mathematics, as perception is for physics, we need a theory of it. After all, we know a great deal about the biological origins of our capacities of perception—but where does mathematical intuition come from?

[1] For other critiques of the indispensability argument, see Maddy's later paper [11] and Feferman [8].

3. The neurological basis of mathematical intuition

> *Intuition implies the act of grasping the meaning or significance or structure of a problem without explicit reliance on the analytic apparatus of one's own craft. It is the intuitive mode that yields hypotheses quickly, that produces interesting comparisons of ideas before their worth is known. It precedes proof; indeed, it is what the techniques of analysis and proof are designed to test and check.*
>
> —J. S. Bruner

> *After all, a mathematical theory that has become the basis of a successful and powerful scientific system, including many important empirical observations, is not being accepted* merely *because it is "intuitive"...*
>
> —Hilary Putnam

The second of the five chapters in this book is "Perception and Intuition". The main issue is that for traditional Platonism, mathematical objects are abstract: How then is it possible for us to know things about them? Here we shift from the problem of ontology—what mathematical things exist?—to that of epistemology—how do we know mathematical truths? "The Platonist still owes us an explanation of how and why Solovay's beliefs about sets are reliable indicators of the truth about sets."

Maddy quotes the nominalist[2] Hartry Field, who denies that mathematics is indispensable for physics. According to strict Platonism, mathematical entities bear no spatiotemporal relation to us, nor do they undergo any physical interactions with us or with anything we can observe; they are mind-independent and language-independent. Then how is it, Field asks, that "our beliefs about these remote entities can so well reflect the facts about them?... *If it appears in principle impossible to explain this*, then that tends to *undermine* the belief in mathematical entities *despite* whatever reason we might have for believing in them... ."

In place of the strict Platonist's unworldly characterization of mathematical objects, Maddy promises to "bring them into the world we know and into contact with our familiar cognitive apparatus."

There follows a long digression on the philosophy, psychology, and neurology of perception, aimed at justifying her central claim, namely, that "We can and do perceive sets, and that our ability to do so develops in much the same way as our ability to perceive physical objects." As a physiological basis for perception, Maddy relies heavily on the neurological speculations of D. O. Hebb in his 1949 book, *The organization of behavior*. Hebb suggested that learning, memory, pattern recognition, and other cognitive tasks are accomplished by modification of structure in the nervous system. In a famous passage he postulated:

> When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process of metabolic change takes place in one or both cells such that A's efficiency, as one of the cells firing B, is increased.

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[2] A nominalist does not believe in the objective existence of mathematical entities.

Hebb suggested that this results in the formation of a *cell assembly*, an interconnected, self-reinforcing set of neurons. By its capacity to respond reliably in the future to the same stimuli that caused it to form originally, the cell assembly is a representation in the nervous system of part of the outside world. Complex perceptions and thoughts correspond to—or simply are—the simultaneous activity of multiple-cell assemblies.[3]

Hebb also suggested there are higher-order assemblies of cell assemblies. Maddy proposes that in our nervous systems there are higher-order cell assemblies corresponding to particular sets, while an even higher-order one corresponds to our general notion of set:

> The structure of this general set assembly is then responsible for various intuitive beliefs about sets, for example, that they have number properties, that those number properties don't change when elements are moved... And these intuitions underlie the most basic axioms of our scientific theory of sets.

This is the epistemological basis for Maddy's "set-theoretic realism". According to this view, our concepts and beliefs about sets come not from Platonic ideal forms in some incomprehensible way, but from certain physical events—changes in synapses and the development of pathways in nervous systems. Similar accounts, she suggests, can be given for lines, curves, and other continuous and geometrical mathematical structures.

4. Numbers

> *It seems to me that the integers have an existence outside ourselves which they impose with the same predetermined necessity as sodium or potassium.*
>
> —C. Hermite
>
> *What is a number, that a man may know it, and a man, that he may know a number?*
>
> —Warren McCulloch

In Chapter 3 Maddy turns her attention to numbers. It is common in formal treatments of set theory to identify numbers with particlar sets. Zermelo identified the natural numbers with the sequence $0, \{0\}, \{\{0\}\}, \ldots$ while von Neumann used $0, \{0\}, \{0, \{0\}\}, \ldots$. Surely it cannot matter which we choose? Perhaps not for mathematical purposes. But if there is no natural choice, then neither choice can be a satisfactory philosophical foundation for the concept of number. For example, the two sequences have different set-theoretic properties: each of Zermelo's numbers after the first is a singleton; not so with von Neumann's. One may object that such properties of these sequences are superfluous—which is precisely the philosophical problem: Why should num-

[3] Far-reaching developments of Hebb's ideas are found in the maverick branch of artificial intelligence called neural networks. For a survey see Anderson and Rosenfeld [1] and Anderson, Pellionisz, and Rosenfeld [2].

bers have superfluous properties? Similar considerations suggest that we really do not want to identify real numbers with sets either.

This argument, due to Benecerraf [4], ultimately shows that not only are numbers not sets, they are not *objects* of any kind: objects lack precisely the universality numbers should have.

Frege considered numbers to be concepts, but Maddy argues against this. And while Cantor thought natural numbers are separate entities "abstracted" from sets and Dedekind said that reals are "associated" with cuts, Maddy objects that they did not explain these processes of abstraction or association.

But realism requires that numbers be *something*. Maddy's solution is that *numbers are properties of sets*. Just as mass, for example, is one of the properties of physical objects that is studied in physics, so also "number" is one of the properties of sets that is studied in mathematics. Just as physical objects are comparable in terms of mass, so sets are comparable in terms of number:

> The von Neumann ordinals are nothing more than a measuring rod against which sets are compared for numerical size. We learn about numbers by learning about the von Neumann ordinals because they form a canonical sequence that exemplifies the properties that numbers have. The choice between the von Neumann ordinals and the Zermelo ordinals is no more than the choice between two different rulers that both measure in metres.

If natural numbers are properties of sets, what are real numbers? Maddy's answer is less perspicuous. She first points out that the question What are the real numbers? is not as directly analogous to What are the natural numbers? as it at first seems. For while von Neumann's and Zermelo's models for the natural numbers have different superfluous set theoretic properties,

> there is after all a single underlying property that all set theoretic versions of the reals serve to detect, a single property shared by all the particular disparate phenomena they are used to measure, namely, continuity. Thus, if there is a proper answer to "what are the reals?" ... then that proper answer is: the real numbers are the property of continuity.

Admittedly "this sounds odd" and rather different from the conclusion that natural numbers are properties of sets. The root of the difference is this. We have quite basic intuitions about naturals, which long predated formal treatments. We do not have such intuitions about reals, but we do have them about *continuity*, which is the concept that needs to be explicated. That is why we cooked up the reals. This much seems clear, but the ontological status of real numbers is left murky.

Included in this chapter is a discussion of *properties*, an intermediate category lying "somewhere between predicates—individuated by sameness of meaning—and sets—individuated by sameness of membership ... ." The chapter concludes with discussions of Frege numbers, which are "collections which are not sets", and of the distinction between sets and classes. While these ideas are not needed for Maddy's development of mathematical realism, they are relevant to her historical account of set theory in the next chapter.

5. Axioms

> *The axiomatization and algebraization of mathematics, after more than fifty years, has led to the illegibility of such a large number of mathematical texts that the threat of complete loss of contact with physics and the natural sciences has been realized.*
>
> —V. I. Arnold
>
> *All this arguing of infinities is but the ambition of schoolboys.*
>
> —Thomas Hobbes

Gödelian Platonism rests on two principles. Maddy explores the first, that the reality of elementary mathematics is justified by our mathematical intuition, in Chapter 3. In Chapter 4 she turns to the second: the justification of less intuitive axioms by their explanatory power.

Before tackling axiomatics, she succinctly recounts the mathematical problem that led Cantor to set theory: the description of the set of points in the line where a Fourier series does not converge. This leads her to Cantor's correspondence with Dedekind and to the set-theoretic hierarchies of Borel, Lebesgue, Baire, Luzin, and Suslin. Reading this, we see clearly how mathematical problems drove the axiomatics.

After preparing the historical scene, Maddy turns to the controversy over the Axiom of Choice that erupted in the first decade of this century, looking at it from the perspective of Compromise Platonism:

> Our best theory of the world requires arithmetic and analysis, and our best theory of arithmetic and analysis requires set theory with at least the axiom of dependent choice. Beyond this pure Quine/Putnamism, the compromise Platonist finds the sort of intra-mathematical arguments that Gödel anticipates.

The Axiom of Choice was justified by many mathematicians on the grounds that mathematics needs it and that it simplified many proofs. Maddy points out the irony that Baire, Borel, and Lebesgue, who were strongly opposed to the new axiom, had actually unwittingly used forms of it many times. Thus their work lent strength to the indispensability argument for Choice put forth by Zermelo.

The fierce debate over the legitimacy of Choice concerned a conflict between two different conceptions of a set: On one side was Frege's logical approach, based on the extension of a concept and the division of everything into two groups according to any kind of rule. On the other side was Cantor's mathematical approach whereby new sets are formed from existing ones according to definite procedures, culminating in Zermelo's iterative hierarchy of sets. We might call these respectively the "top down" and "bottom up" approaches. Baire, Borel, and Lebesgue, suspicious of arbitrary correspondences, used the bottom up approach to treat functions.

Maddy quotes from a series of letters between these three analysts and their opponent, Hadamard. In 1905, following Zermelo's use of Choice to prove the well-ordering principle, Lebesgue wrote to Hadamard:

> The question comes down to this, which is hardly new: *Can one prove the existence of a mathematical object without defining it?* .. I believe that we can only build solidly *by granting that it is impossible to demonstrate the existence of an object without defining it.*

Who could object to such a reasonable principle? Hadamard could! Admitting that Zermelo had no way of carrying out the mapping needed for a choice function, he insisted that the problem of its effective determination is completely distinct from the question of its existence: "The existence... is a fact like any other." Today Hadamard's pro-Choice position has prevailed with the vast majority of mathematicians.

Next in this chapter comes a discussion of some open problems in set theory, such as the Continuum Hypothesis (which Cantor believed, but Gödel did not). Maddy surveys the prospects of settling such problems by several "competing theories", obtained by adding various new axioms to the standard ones.

After a rather technical discussion of what is provable under which axioms, there is an interesting statement of why Compromise Realism leads the author to "avoid attributing intuitive status" to certain motivations that have been put forth as justification for the Axiom of Large Cardinals: The reason, Maddy says, is "because I think they extend beyond anything that could plausibly be traced to an underlying perceptual, neurological foundation... ."

After looking at some of the competing axiom systems for set theory, not all of which can be true since they lead to opposite determinations of the Continuum Hypothesis, Maddy poses the challenge of deciding which of them is more likely to be true. This is not a question that can be settled by a formal proof, since the very problem is to choose the axioms on which to base proofs. And we cannot rely on the old idea that we accept only axioms that are self-evident, since even the accepted axioms of ZFC (Zermelo-Fraenkel set theory with the Axiom of Choice) "do not enjoy this status":

> A new account of our knowledge of axioms and of the evidential role of nondemonstrative mathematical arguments in general is clearly needed.... But before we can answer the question of which axiom candidate is supported by better such arguments, we must face the prior question of whether these arguments carry any weight at all, and if so why. We need to explain how, why, and to what extent such arguments count as evidence for the truth of their conclusions. Only then can we determine which among them constitute the better evidence.

The final chapter returns to the defense of set-theoretic realism, quoting and answering attacks by the nominalists H. Field and C. Chihara, and comparing the author's Platonism to monism, physicalism, and structuralism. The author demonstrates that these competing philosophies share with mathematical realism the problem of evaluating the claims of different axiom systems by nondemonstrative methods.

Finally there is an admirably clear and succinct summary of her views.

6. Mathematical truth

> *I believe there are exactly* 15, 747, 724, 136, 275, 002, 577, 605, 653, 961, 181, 555, 468, 044, 717, 914, 527, 116, 709, 366, 231, 425, 076, 185, 631, 031, 296 *protons in the universe, and the same number of electrons.*
>
> —Sir Arthur Eddington

One might expect a book of this sort to grapple with the problem of mathematical truth: Does every mathematical statement have a truth value? If so, what does it mean to say that a mathematical statement, for which there currently exists no proof, is "true"?

Unless we somehow tie mathematical abstractions to the physical world, these questions are answerable only as matters of faith. But for the mathematical realist, mathematical truth raises no special difficulties: Mathematics is about things that "exist and are as they are independently of our ability to know about them." A mathematical statement about, say, real numbers is either true or false, and our task is to discover which—but our success or failure at this task does not affect the truth value. A mathematical statement is true if and only if it corresponds to mathematical reality, and that is all there is to it. It is not that the problem of mathematical truth does not exist—but it is part of the larger philosophical problem: which statements are true?

According to set-theoretic realism, there is a fact of the matter about, for example, the Continuum Hypothesis: Either it is true or it is false. Since intuition does not help much, the realist's task is to search for an axiom system that will lead to the truth.

There is another consideration, however, which is not discussed in this book. A proposition which seems to be meaningful today may eventually, as science develops, come to be viewed as unanswerable in principle (and thus scientifically meaningless), or to be based on a false assumption about reality, or to be just irrelevant. Whole research programs disappear in this way.

This has happened frequently in many branches of science. A century ago the most important question in biology was to discover the nature of the life force; in physics, to discover the nature of the ether. It is no longer meaningful to ask about the simultaneous position and momentum of an electron, or about simultaneous events in distant galaxies, or about precisely which radium atom will decay next. At one time our best science said there were exactly five planets; then there were six .... Sometimes the opposite occurs, as when the alchemists' obsolete dream of transmutation of elements was realized by radioactive decay and nuclear explosion.

Similar things have happened in mathematics:

- Before Pythagoras it was a mathematical truth that every ratio is rational.
- Consider the changing truth value of Euclid's parallel postulate over the centuries, bearing in mind that not so long ago, geometry was not merely an axiomatic study, but our best scientific description of physical space.
- From ancient times through the seventeenth century, there was lively discussion of whether the line is composed of infinitesimals or indivisibles.

- Gödel's incompleteness theorems aborted Hilbert's program of proving mathematics complete and consistent.
- Infinitesimals, discredited in mathematics for a century, were exonerated by A. Robinson's invention (discovery?) of nonstandard analysis.

We are of course free to believe a statement currently deemed scientifically or mathematically meaningless, if we find this comforting; but this is a private act, incapable (currently!) of scientific justification—analogous to attributing truth value to "The soul is located in the pineal gland." But one who wants to communicate such a belief must explain what it means. If you tell me that it is true, although not currently provable, that for every n there exist $n!$ consecutive 7s in the decimal expansion of π, then you have to explain your use of "true". Such explanations used to refer to the mind of God, but eternally running computers are now more fashionable.

What could it mean that, at some future time, the Continuum Hypothesis is judged, by our best scientific and mathematical theory, to be meaningless or irrelevant? This is of course hard to say, since such a consensus would be based on some as-yet nonexistent new knowledge. But here are some scenarios:

(a) Some disturbing new kind of set-theoretic anomalies are discovered, which convince us that some form of constructivism is absolutely essential for making sense of sets.

(b) Neurologists and psychologists learn enough about cell assemblies and cognition to make it scientifically certain that there could not possibly be any activity in the nervous system which would correspond to a truth value for the Continuum Hypothesis.

(c) Some kind of higher-level, theoretical, or extrinsic justifications might convince us that the Continuum Hypothesis conflicts with widely accepted views about mathematical reality.

My inclination is toward (b). Even in the absence of such biological discoveries, I think it is highly likely that our ability to make mathematical definitions has already far outrun the capabilities of our cell assemblies to discover a fact of the matter in the Continuum Hypothesis.

To the mathematical realist, truth about mathematical entities is just as problematical as truth about physical objects—but no more so. Our current mathematical theories, like our physical theories, are approximately correct; but there is no reason to think they are infallible or that every question which is meaningful today will always remain so.

7. Conclusions

> *I do not think that the difficulties that philosophy finds with classical mathematics today are genuine difficulties; and I think that the philosophical interepretations of mathematics that we are being offered on every hand are wrong, and that "philosophical interpretation" is just what mathematics doesn't need.*
>
> —Hilary Putnam

Professor Maddy presents a vigorous, clearly written philosophical foundation for the working mathematician's intuitive feeling that mathematics is about real things. Accepting the Quine/Putnam argument that at least some mathematical entities are real because they are indispensable for physics, she also adopts Gödel's thesis that we have a faculty of mathematical intuition, basing it on speculative but plausible neurological mechanisms.

In her view, we have the capacity to perceive not only individual physical objects but also sets of them, *as* sets. Changes in our neural pathways give us the power to form intuitions about sets and elementary operations on them, analogous to our intuitions about lengths and other properties of physical objects. Cardinal numbers are not sets but properties of sets. The real number system is a setting constructed to explicate our intuitions of continuity.

Intuition is no longer an adequate justification for mathematical realism once we get into the more theoretical parts of set theory. Here we need "nondemonstrative arguments" to justify recondite axioms or to decide between two axiom systems on the basis of the evidence for their plausibility in terms of their provable consequences. Maddy concludes:

> A modest contribution to that project is all that has been attempted here. The next step, the evaluation of this evidence, is a daunting undertaking, but I've argued that the set theoretic realist faces this challenge in the distinguished company of thinkers representing a wide range of competing mathematical philosophies, structuralism, modalism, and a version of nominalism among them.

For the philosophically inclined mathematician, I highly recommend this thoughtful, provocative book.

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Introduction to regularity theory for nonlinear elliptic systems, by Mariano Giaquinta. Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1993, viii + 130 pp., \$29.00. ISBN 3-7643-2879-7

True mathematical understanding of nature is impossible without an understanding of the partial differential equations and variational principles that govern a large part of physics. Already very early in the development of calculus, besides the linear equations of electrostatics, for example, also nonlinear partial differential equations were studied. A prominent example is the nonparametric minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right)=0, \qquad u\colon \Omega\subset\mathbb{R}^2\to\mathbb{R},$$

which, in 1762, Lagrange derived as an illustration of what later became the Euler-Lagrange variational principle. The expression

$$\operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$$

(up to a factor) gives the mean curvature of the hypersurface $S\subset\mathbb{R}^3$ defined by the graph of u.

Looking back, it is no surprise that nonlinear partial differential equations first arose from an interplay of physics and geometry. In the eighteenth century, however, such a distinction would have been meaningless, as mathematics and physics were still largely being considered as a whole. Today many mathematicians and theoretical physicists are turning back to this view, largely because more and more examples emerge of nonlinear partial differential equations that play a fundamental role both in geometry and in physics: Harmonic maps, Yang-Mills equations, Einstein equations, etc.

Very often such equations arise from minimization problems

$$E(u)=\int_\Omega f(x, u(x), \nabla u(x))\,dx\to\min, \tag{0.1}$$

where admissible comparison functions $u=(u^1, \dots, u^N)\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$ may be constrained, for instance, by boundary conditions, and where the function $f\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^{n\cdot N}\to\mathbb{R}$ is smooth in all its variables, bounded from below, and