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A SPLITTING PROPERTY FOR SUBALGEBRAS OF TENSOR PRODUCTS

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ABSTRACT. We prove a basic result about tensor products of a II_1 factor with a finite von Neumann algebra and use it to answer, affirmatively, a question asked by S. Popa about maximal injective factors.

1. INTRODUCTION

The splitting property referred to in the title states that, in certain situations, a subalgebra $\mathscr{A}$ of the tensor product $\mathscr{A}_1 \otimes \mathscr{A}_2$ of two algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ (with units) that contains $\mathscr{A}_1$ "splits" as a tensor product $\mathscr{A}_1 \otimes \mathscr{A}_0$, where $\mathscr{A}_0$ is a subalgebra of $\mathscr{A}_2$. If $\mathscr{A}_1$ is $M_n(\mathbb{C})$, the algebra of all $n \times n$ matrices over the complex numbers $\mathbb{C}$, and $\mathscr{A}_2$ is another complex matrix algebra, with the tensor product taken over $\mathbb{C}$, the splitting results from a simple algebraic calculation. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are arbitrary, infinite-dimensional algebras over fields of arbitrary characteristic, with varying assumptions on their structure, the situation is less clear.

When $\mathscr{A}_1$ and $\mathscr{A}_2$ have topological and analytic structure and the tensor product $\mathscr{A}_1 \otimes \mathscr{A}_2$ is formed to reflect that structure, the splitting question becomes a deeper and more intricate one. The principal result of this note involves von Neumann algebras $\mathscr{R}_1$ and $\mathscr{R}_2$ and their von Neumann-algebra tensor product $\mathscr{R}_1 \bar{\otimes} \mathscr{R}_2$. Specifically, we prove the following result.

Theorem A. *If $\mathscr{M}$ is a factor of finite type, $\mathscr{R}$ is a finite von Neumann algebra, and $\mathscr{S}$ is a von Neumann subalgebra of $\mathscr{M} \bar{\otimes} \mathscr{R}$ that contains $\mathscr{M}$ ($= \mathscr{M} \bar{\otimes} \mathbb{C}I$), there is a von Neumann subalgebra $\mathscr{R}_0$ of $\mathscr{R}$ such that $\mathscr{S} = \mathscr{M} \bar{\otimes} \mathscr{R}_0$.*

In the preceding statement, we use "finite" in the sense of Murray and von Neumann [M-vN]: $\mathscr{M}$ is either a factor of type II_1 or of type I_n (isomorphic to $M_n(\mathbb{C})$).

Using Theorem A, we answer, affirmatively, a question raised by Sorin Popa [P1]. Recall that an *injective* von Neumann algebra acting on a Hilbert space $\mathscr{H}$ is one that is the image of an idempotent, norm 1 mapping of $\mathscr{B}(\mathscr{H})$, the algebra of all bounded operators on $\mathscr{H}$. Popa asks [P1, Problem 4.5(1)]:

If $\mathscr{M}_1$, $\mathscr{M}_2$ are type II_1 factors and $\mathscr{B}_1 \subset \mathscr{M}_1$, $\mathscr{B}_2 \subset \mathscr{M}_2$ are maximal injective von Neumann subalgebras, is $\mathscr{B}_1 \otimes \mathscr{B}_2$ maximal injective in $\mathscr{M}_1 \otimes \mathscr{M}_2$? Is this true at least for $\mathscr{M}_2 = \mathscr{B}_2 = \mathscr{R}$, where $\mathscr{R}$ is the hyperfinite II_1 factor?

We show that the answer is affirmative in the case where $\mathscr{R}$ is the hyperfinite II_1 factor.

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Further work is in progress, by the author and R. Kadison, on the splitting property when the restriction on the types of the von Neumann algebras is removed.

2. Methods and preliminary results

We make important use of the slice-map technique of Tomiyama [To] (especially as formulated in [K-R IV, Exercise 12.4.36]). Conditional expectation techniques and Schwartz projections [Sc] play a key role in the arguments. A description of these techniques follows.

From [K-R IV], let $\mathscr{R}$ and $\mathscr{S}$ be von Neumann algebras and ρ and σ be non-zero elements of $\mathscr{R}_\sharp$ and $\mathscr{S}_\sharp$, respectively. Then

(i) there is a unique element $\rho\otimes\sigma$ of $(\mathscr{R}\bar{\otimes}\mathscr{S})_\sharp$ such that $(\rho\otimes\sigma)(R\otimes S)=\rho(R)\sigma(S)$ for each R in $\mathscr{R}$ and S in $\mathscr{S}$ and that $\|\rho\otimes\sigma\|=\|\rho\|\|\sigma\|$;

(ii) there are unique operators $\Phi_\sigma(\tilde{T})$ and $\Psi_\rho(\tilde{T})$ in $\mathscr{R}$ and $\mathscr{S}$, respectively, corresponding to each $\tilde{T}$ in $\mathscr{R}\bar{\otimes}\mathscr{S}$, satisfying

$$\rho'(\Phi_\sigma(\tilde{T}))=(\rho'\otimes\sigma)(\tilde{T}), \qquad \sigma'(\Psi_\rho(\tilde{T}))=(\rho\otimes\sigma')(\tilde{T})$$

for each ρ' in $\mathscr{R}_\sharp$ and each σ' in $\mathscr{S}_\sharp$;

(iii) Φ_σ and Ψ_ρ (as defined by (ii)) are ultraweakly continuous linear mappings of $\mathscr{R}\bar{\otimes}\mathscr{S}$ onto $\mathscr{R}$ and $\mathscr{S}$, respectively, satisfying

$$\Phi_\sigma((A\otimes I)\tilde{T}(B\otimes I))=A\Phi_\sigma(\tilde{T})B,$$
$$\Psi_\rho((I\otimes C)\tilde{T}(I\otimes D))=C\Psi_\rho(\tilde{T})D$$

for each $\tilde{T}$ in $\mathscr{R}\bar{\otimes}\mathscr{S}$; A, B in $\mathscr{R}$; and C, D in $\mathscr{S}$; and

$$\Phi_\sigma(R\otimes S)=\sigma(S)R, \qquad \Psi_\rho(R\otimes S)=\rho(R)S$$

when $R\in\mathscr{R}$ and $S\in\mathscr{S}$;

(iv) $\Phi_\sigma(\tilde{T})\in\mathscr{R}_0$ and $\Psi_\rho(\tilde{T})\in\mathscr{S}_0$ if $\tilde{T}\in\mathscr{R}_0\bar{\otimes}\mathscr{S}_0$, where $\mathscr{R}_0$ and $\mathscr{S}_0$ are von Neumann subalgebras of $\mathscr{R}$ and $\mathscr{S}$, respectively;

(v) $\tilde{T}\in\mathscr{R}_0\bar{\otimes}\mathscr{S}_0$ if $\Phi_{\sigma'}(\tilde{T})\in\mathscr{R}_0$ and $\Psi_{\rho'}(\tilde{T})\in\mathscr{S}_0$ for each σ' in $\mathscr{S}_\sharp$ and each ρ' in $\mathscr{R}_\sharp$;

(vi) $\Phi_\sigma(\tilde{T})$ appears, naturally, as a bounded linear functional on $\mathscr{R}_\sharp$ (thus, it is an element of $\mathscr{R}$). It is defined by

$$\Phi_\sigma(\tilde{T})\colon \rho'\mapsto(\rho'\otimes\sigma)(\tilde{T}) \qquad (\rho'\in\mathscr{R}_\sharp)$$

and

$$\begin{aligned}|(\Phi_\sigma(\tilde{T}))(\rho')|&=|(\rho'\otimes\sigma)(\tilde{T})|\\&\le\|\rho'\|\|\sigma\|\|\tilde{T}\|.\end{aligned}$$

From this, (v) is valid under the assumption that σ' and ρ' may be any elements of families of normal states that generate, linearly, norm-dense subspaces of $\mathscr{S}$ and $\mathscr{R}$, respectively.

Suppose $\mathscr{R}$ is the ultraweak closure of an ascending union of finite-dimensional C*-algebras $\mathfrak{A}_n$ acting on a Hilbert space $\mathscr{H}$ (we call this union a *tower* for $\mathscr{R}$). Each $\mathfrak{A}_n$ has a finite group $\mathscr{U}_n$ of unitary elements that generates it linearly. In [Sc], Schwartz constructs a linear mapping of $\mathscr{B}(\mathscr{H})$ onto the commutant $\mathscr{R}'$ of $\mathscr{R}$ (a *Schwartz projection*) by averaging over the groups $\mathscr{U}_n$, successively, and passing to a Banach limit.

Given a tower t for $\mathscr{R}$ and a free ultrafilter p, there is a norm 1, linear, idempotent mapping (a *conditional expectation*) $\Phi_{t,p}$ of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{R}'$ such that

Proposition B. *Each* $\Phi_{t,p}$ *is a conditional expectation of* $\mathscr{B}(\mathscr{H})$ *onto* $\mathscr{R}'$ *proper on each von Neumann algebra* $\mathscr{S}$ *containing* $\mathscr{R}$ (*onto* $\mathscr{S}\cap\mathscr{R}'$).

For this proposition, we need

Definition C. A conditional expectation Ψ of a von Neumann algebra $\mathscr{S}$ onto a subalgebra is *proper* when $\Psi(T)\in \mathrm{co}_{\mathscr{S}}(T)^-$ for each T in $\mathscr{S}$, where $\mathrm{co}_{\mathscr{S}}(T)^-$ is the weak-operator closure of $\mathrm{co}_{\mathscr{S}}(T)$ and

$$\mathrm{co}_{\mathscr{S}}(T)=\{UTU^* : U \text{ a unitary in } \mathscr{S}\}.$$

In the next result, we describe a conditional expectation Φ of a finite von Neumann algebra $\mathscr{M}$ onto a von Neumann subalgebra $\mathscr{N}$ and describe some of its properties that will be needed.

Theorem D. *If* $\mathscr{N}$ *is a von Neumann subalgebra of a finite von Neumann algebra* $\mathscr{M}$ *with* (*normalized*) *trace* τ, *there is a conditional expectation* Φ *of* $\mathscr{M}$ *onto* $\mathscr{N}$, *whose trace* $\tau|\mathscr{N}$ *we denote by* "τ_0", *such that*

$$\tau(TA)=\tau_0(\Phi(T)A)\qquad (T\in\mathscr{M},\ A\in\mathscr{N}).$$

Moreover, Φ *is the unique conditional expectation of* $\mathscr{M}$ *onto* $\mathscr{N}$ *such that*

$$(*)\qquad \tau(T)=\tau_0(\Phi(T))\qquad (T\in\mathscr{M}).$$

If $\mathscr{M}$ *admits a proper conditional expectation* Ψ *of* $\mathscr{M}$ *onto* $\mathscr{N}$, *then* $\Psi=\Phi$.

3. An outline of the proof

We use the notation of Theorem A and Section 2. We assume that $\mathscr{M}$ is a factor of type II_1. If $\mathscr{S}$ is a von Neumann subalgebra of $\mathscr{M}\bar{\otimes}\mathscr{R}$ containing $\mathscr{M}\bar{\otimes}\mathbb{C}I$ and $\mathscr{T}=\mathscr{M}'\cap\mathscr{S}$, what we want to show is that $\mathscr{M}\bar{\otimes}\mathscr{T}=\mathscr{S}$. Since $\mathscr{M}\bar{\otimes}\mathbb{C}I\subseteq\mathscr{S}$ and $\mathscr{T}\subseteq\mathscr{S}$, of course $\mathscr{M}\bar{\otimes}\mathscr{T}\subseteq\mathscr{S}$. Now we use the slice-map technique. If $T\in\mathscr{S}$, since Φ_σ maps onto $\mathscr{M}$, of course $\Phi_\sigma(T)\in\mathscr{M}$ for each σ. Define τ_H on $\mathscr{M}$ by $\tau_H(M)=\tau(HM)$ when $H, M\in\mathscr{M}$. We know that $\{\tau_H : H\in\mathscr{M}\}$ generate a norm-dense subspace of $\mathscr{M}_\sharp$ [K-R II, Theorem 7.2.3]. It remains to show that

$$\Psi_{\tau_H}(T)=\Psi_\tau((H\otimes I)T)\in\mathscr{T}\qquad (H\in\mathscr{M}).$$

But Ψ_τ lifts the trace from $\mathscr{R}$ to $\mathscr{M}\bar{\otimes}\mathscr{R}$, so Ψ_τ is the unique conditional expectation of $\mathscr{M}\bar{\otimes}\mathscr{R}$ onto $\mathscr{R}$ satisfying $(*)$ of Theorem D.

By a deep result of S. Popa (see [P2]), there is a hyperfinite subfactor $\mathscr{R}_0$ of $\mathscr{M}$ with trivial relative commutant. Since $\mathscr{R}_0$ is hyperfinite, each $\Phi_{t,p}$ (for $\mathscr{R}_0$) is proper, and the restriction of $\Phi_{t,p}$ to $\mathscr{M}\bar{\otimes}\mathscr{R}$ coincides with Ψ_τ (from Theorem D). Now $\mathrm{co}_{\mathscr{R}_0}((H\otimes I)T)\subseteq\mathscr{S}$—recall that $T\in\mathscr{S}$ and $H\otimes I\in\mathscr{M}\subseteq\mathscr{S}$. So

$$\Psi_{\tau_H}(T)=\Psi_\tau((H\otimes I)T)=\Phi_{t,p}((H\otimes I)T)\in\mathrm{co}_{\mathscr{R}_0}((H\otimes I)T)^-\subseteq\mathscr{S}.$$

But $\Phi_{t,p}$ maps onto $\mathscr{R}_0'$, so $\Phi_{t,p}((H\otimes I)T)\in\mathscr{R}_0'\cap\mathscr{S}=\mathscr{M}'\cap\mathscr{S}=\mathscr{T}$ [K-R II, IV, 12.4.37]. This completes the argument.

The corollary that follows answers the Popa question noted in the introduction, even in a slightly generalized form. The proof of it follows directly from our Theorem A and properties of injective von Neumann algebras.

Corollary E. *Let $\mathscr{R}$ be the hyperfinite II_1 factor and $\mathscr{S}$ a type II_1 von Neumann algebra with a maximal injective von Neumann subalgebra $\mathscr{B}$. Then $\mathscr{R}\,\bar{\otimes}\,\mathscr{B}$ is maximal injective in $\mathscr{R}\,\bar{\otimes}\,\mathscr{S}$.*

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