

SECONDARY INVARIANTS AND THE SINGULARITY OF THE RUELLE ZETA-FUNCTION IN THE CENTRAL CRITICAL POINT

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ABSTRACT. The Ruelle zeta-function of the geodesic flow on the sphere bundle $S(X)$ of an even-dimensional compact locally symmetric space X of rank 1 is a meromorphic function in the complex plane that satisfies a functional equation relating its values in s and $-s$. The multiplicity of its singularity in the central critical point $s = 0$ only depends on the hyperbolic structure of the flow and can be calculated by integrating a secondary characteristic class canonically associated to the flow-invariant foliations of $S(X)$ for which a representing differential form is given.

Let $Y = G/K$ be a rank one symmetric space of the non-compact type, i.e., Y is a real, complex, or quaternionic hyperbolic space or the (16-dimensional) hyperbolic Cayley-plane. Let Γ be a uniform lattice in the (connected simple) isometry group G of Y without torsion. Γ acts properly discontinuous on $Y = G/K$ (K a maximal compact subgroup of G), and $X = \Gamma \backslash G/K$ is a compact locally symmetric space. We consider X as a Riemannian manifold with respect to an arbitrary (constant) multiple g of the metric g_0 induced by the Killing form on the Lie algebra of G . Then the Riemannian manifold (X, g) is a space of negative curvature.

The negativity of the curvature of the metric g on X implies the existence of an infinite countable set of prime closed geodesics in X with a discrete set of prime periods accumulating at infinity.

Let Φ_t be the geodesic flow on the unit sphere bundle $S(X)$ of the space (X, g) .

The prime period of a periodic orbit of Φ_t on $S(X)$ coincides with the length of the closed geodesic in X obtained by projecting the periodic orbit into X . Now we use these periods to define the zeta-function

$$(1) \quad Z_R(s) = \prod_c (1 - \exp(-sl_c))^{-1}$$

for $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > h$ (h being the topological entropy of the geodesic flow Φ_t on $S(X)$). The product in (1) runs over all closed *oriented* geodesics c in X , and l_c denotes the length of c as a curve in X . Note that for each (unoriented) closed geodesic c in X there are two lifts of c as periodic orbits of Φ_t in correspondence with the two possibilities to orient c .

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The function Z_R is well known as the Ruelle zeta-function of the geodesic flow [5]. The Euler product (1) defines a holomorphic function in the half-plane $\text{Re}(s) > h$.

Now by symbolic dynamics the zeta-function Z_R can be written for large $\text{Re}(s)$ as an alternating product of zeta-functions associated to suspensions of subshifts of finite type. The latter zeta-functions coincide with Fredholm-determinants $\det(1 - L_*(s))$ (in the sense of Grothendieck) of holomorphic families $L_*(s)$ of nuclear transfer operators on certain spaces of differential forms. This representation implies that Z_R has a meromorphic continuation to the complex plane (see [5], [14]). While these arguments establish the existence of a meromorphic continuation, it seems to be rather difficult to prove results on the positions and the multiplicities of the singularities of Z_R by the same method.

On the other hand, the zeta-function Z_R can be written as a product of generalized Selberg zeta-functions. Generalized Selberg zeta-functions are also defined by Euler products similar to (1) but with more complicated local Euler factors containing monodromy contributions of the loops c in certain vector bundles on $S(X)$ (see [3], [5], [7], [12], [17]). The generalized Selberg zeta-functions in turn can be investigated by using trace formula techniques which are at present the only known methods to uncover the deeper relations between periodic orbits and the geometry and topology of the underlying space. However, in contrast to the dynamical point of view, along the usual trace formula arguments the relation between the hyperbolic structure and the analytical properties of the zeta-function remains mysterious.

The main result of the present note describes a direct relation between the multiplicity of the singularity of Z_R in the central critical point $s = 0$ and the hyperbolic structure of the flow.

Let us first consider the well-known special case of the Ruelle zeta-function of the geodesic flow of a compact Riemannian surface $X = \Gamma \backslash H^2$ of negative Euler characteristic $\chi(X)$. Consider X as a Riemannian manifold with the metric inherited from the hyperbolic metric

$$(2) \quad y^{-2}(dx^2 + dy^2)$$

of constant curvature -1 on the upper half plane H^2 . Then the product (1) defines a holomorphic function in the half-plane $\text{Re}(s) > 1$. Moreover, $Z_R(s)$ satisfies the functional equation

$$(3) \quad Z_R(s)Z_R(-s) = ((1 - \exp(2\pi is))(1 - \exp(-2\pi is))^{2-2g}.$$

In particular, for the multiplicity m_0 of its singularity in $s = 0$ we have

$$(4) \quad m_0 = 2 - 2g = \chi(X).$$

The functional equation (3) is a consequence of the functional equation

$$Z_S(1 - s) = Z_S(s) \exp \left(2(2 - 2g) \int_0^{s-1/2} (\pi t) \tan(\pi t) dt \right),$$

for the classical Selberg zeta-function

$$(5) \quad Z_S(s) = \prod_c \prod_{N \geq 0} (1 - \exp(-(s + N)l_c)), \quad \operatorname{Re}(s) > 1, \quad s \in \mathbb{C},$$

(see [16]) and the obvious relation $Z_R(s) = Z_S(s + 1)/Z_S(s)$.

The Ruelle zeta-function Z_R always satisfies a functional equation similar to (3) relating $Z_R(s)$ to $Z_R(-s)$. However, the general theory of the functional equation for Z_R will not be discussed here. Instead we shall only discuss certain formulas for the multiplicity m_0 of the singularity of Z_R in $s = 0$ generalizing formula (4).

To formulate the main result, we shall use group-theoretical descriptions of the geodesic flow and its hyperbolic structure (Anosov property).

Let g_0 and k_0 be the respective Lie algebras of G and K , and let $g_0 = k_0 \oplus p_0$ be the Cartan decomposition of g_0 being orthogonal with respect to the Killing form. Identify p_0 with the tangent space $T_{eK}(Y)$. Let $a_0 \subset p_0$ be a one-dimensional abelian subspace, and let $M \subset K$ be the centralizer of a_0 in K .

Now consider the action

$$\Phi_A : A \times \Gamma \backslash G/M \longrightarrow \Gamma \backslash G/M, \quad A = \exp(a_0),$$

defined by

$$(6) \quad (a, \Gamma gM) \mapsto \Phi_a(\Gamma gM) = \Gamma ga^{-1}M, \quad a \in A.$$

The definition of the action Φ_A is independent of the choice of a metric on X , and we shall denote this action in the following as the *abstract geodesic flow*.

Once and for all we fix an orientation of the flow Φ_A which amounts to fixing an (open) Weyl-chamber a_0^+ in a_0 , called the positive chamber, and we shall restrict attention only to the action of $A^+ = \exp(a_0^+) \subset A$.

Now in terms of the abstract geodesic flow Φ_A on $\Gamma \backslash G/M$ the hyperbolic structure can be described as follows.

Let n_0^+ and n_0^- be the subspaces of g_0 on which $ad(X)$ for $X \in a_0^+$ acts by positive and negative eigenvalues $\alpha(X)$, respectively. The nilpotent Lie algebras n_0^\pm are real MA -modules with respect to the adjoint action.

Let \mathcal{P}^\pm be the locally homogeneous vector bundles

$$(7) \quad \Gamma \backslash G \times_M (n_0^\pm) \rightarrow \Gamma \backslash G/M,$$

and regard these as subbundles of the tangent bundle $T(\Gamma \backslash G/M)$. The real vector bundles \mathcal{P}^\pm are integrable, and the foliation of $\Gamma \backslash G/M$ obtained by integrating \mathcal{P}^+ (\mathcal{P}^-) is the Φ_A -invariant unstable (stable) foliation of $\Gamma \backslash G/M$. More precisely, the tangent bundle $T(\Gamma \backslash G/M)$ admits a $d(\Phi_A)$ -equivariant decomposition

$$(8) \quad T(\Gamma \backslash G/M) = \mathcal{P}^+ \oplus T^0(\Gamma \backslash G/M) \oplus \mathcal{P}^-$$

into the direct sum of the stable subbundle \mathcal{P}^- , the central subbundle $T^0(\Gamma \backslash G/M)$, and the unstable subbundle \mathcal{P}^+ . Tangent vectors in \mathcal{P}^- (\mathcal{P}^+) are contracted (expanded) exponentially by the differential $d(\Phi_a)$, $a \in A^+$.

Note that the stable and the unstable leaves of $\Gamma \backslash G/M$ are smoothly embedded smooth submanifolds.

Now we shall associate to both foliations of $\Gamma \backslash G/M$ canonical differential forms

$$\Omega_R(\mathcal{P}^\pm) \in C^\infty(\Lambda^{2d-2}T^*(\Gamma \backslash G/M)), \quad d = \dim(X).$$

We begin with the construction of left G -invariant and $\text{End}(\hat{\mathcal{P}}^\pm)$ -valued 2-forms

$$\omega_R^\pm \in C^\infty(\Lambda^2T^*(G/MA) \otimes \text{End}(\hat{\mathcal{P}}^\pm)),$$

where $\hat{\mathcal{P}}^\pm$ denotes the G -homogeneous vector bundle

$$G \times_{MA} (n_0^\pm) \rightarrow G/MA.$$

By G -invariance it suffices to define ω_R^\pm in $\underline{e} = eMA$.

We choose a (real) basis $\{Z_j\}$ of the space n_0^\pm . Let $\{Z^j\}$ be the dual basis of $(n_0^\pm)^*$. Set

$$(9) \quad (\omega_R^\pm)_\underline{e} = \left((\omega_R^\pm)_\underline{e}(\circ, \circ) \right)_{j,k}^k, \quad j, k = 1, \dots, \dim(n_0^\pm) = d - 1,$$

where

$$(10) \quad (\omega_R^\pm)_\underline{e}(\hat{X}_\underline{e}, \hat{Y}_\underline{e})^k = \langle Z^k, -[[X, Y]_0, Z_j] \rangle$$

for $X, Y \in n_0^+ \oplus n_0^-$ and $\hat{X}_\underline{e} = d_e(\pi)(X)$ ($\pi : G \rightarrow G/MA$ being the canonical projection). Here $[X, Y]_0$ denotes the $m_0 \oplus a_0$ -component of $[X, Y]$ with respect to the decomposition $\mathfrak{g}_0 = (m_0 \oplus a_0) \oplus n_0^+ \oplus n_0^-$, where m_0 denotes the Lie algebra of M .

Regard the matrix $(\omega_R^\pm)_\underline{e}$ as an $\text{End}(n_0^\pm)$ -valued alternating 2-form on $T_\underline{e}(G/MA)$.

Now extend $(\omega_R^\pm)_\underline{e}$ to a G -invariant and $\text{End}(\hat{\mathcal{P}}^\pm)$ -valued 2-form ω_R^\pm on G/MA .

Next lift ω_R^\pm via $G/M \rightarrow G/MA$ to an $\text{End}(\mathcal{P}^\pm)$ -valued G -invariant differential 2-form on G/M . The latter 2-form drops down to an $\text{End}(\mathcal{P}^\pm)$ -valued 2-form on $\Gamma \backslash G/M$ also denoted by ω_R^\pm .

Now define

$$(11) \quad \Omega_R(\mathcal{P}^\pm) = \det((i/2\pi)\omega_R^\pm).$$

Then the forms $\Omega_R(\mathcal{P}^\pm) \in C^\infty(\Lambda^{2d-2}T^*(\Gamma \backslash G/M))$ are *closed basic* forms with respect to the foliation of $\Gamma \backslash G/M$ by the orbits of the abstract geodesic flow Φ_A , i.e., $\Omega_R(\mathcal{P}^\pm)$ is Φ_A -invariant and $i_{\hat{X}}(\Omega_R(\mathcal{P}^\pm)) = 0$ for all sections \hat{X} of $T^0(\Gamma \backslash G/M)$.

If the dimension of X is even, then it follows that

$$\Omega_R(\mathcal{P}^+) = -\Omega_R(\mathcal{P}^-),$$

and the uniquely determined *real eigenvalue* $\neq 0$ of the $\text{End}(\mathcal{P}^\pm)$ -valued 2-form ω_R^\pm on $\Gamma \backslash G/M$ is a well-defined Φ_A -invariant 2-form

$$\mu_R^\pm \in C^\infty(\Lambda^2T^*(\Gamma \backslash G/M))$$

on $\Gamma \backslash G/M$. Since μ_R^\pm is an *exact* 2-form, it follows that there exists a (uniquely determined) left G -invariant and right Φ_A -invariant 1-form α_R^\pm on G/M such

that α_R^\pm drops down to a Φ_A -invariant 1-form

$$\alpha_R^\pm \in C^\infty(T^*(\Gamma \backslash G/M))$$

which satisfies

$$(12) \quad d\alpha_R^\pm = (i/2\pi)\mu_R^\pm.$$

The main reason to formulate all constructions by using the abstract geodesic flow Φ_A instead of the geodesic flow Φ_t is that since the multiplicity m_0 is independent of the choice of any scaling of the negative curvature metric on X , it is natural to look for a formula for m_0 which does not refer to the metric.

Now let $\phi : S(X) \rightarrow \Gamma \backslash G/M$ be the diffeomorphism obtained by composing the canonical isomorphism of $S(X)$ and $\Gamma \backslash S(Y)$ with the G -equivariant map $S(Y) \ni g(eK, X) \mapsto gM \in G/M$, $X \in a_0^+$ of $S(Y)$ onto G/M . The stable and unstable foliations of $\Gamma \backslash G/M$ then obviously correspond to the stable and unstable foliations of $S(X)$ associated to the geodesic flow on $S(X)$.

Theorem 1. *Let the dimension of X be even. Then the multiplicity m_0 of the singularity of Z_R in $s = 0$ is given by the formula*

$$(13) \quad m_0 = \int_{S(X)} \phi^*(\Omega_R(\mathcal{P}^+) \wedge \alpha_R^-) = \int_{S(X)} \phi^*(\Omega_R(\mathcal{P}^-) \wedge \alpha_R^+).$$

Moreover, the functional equation of Z_R (not given here) implies that all singularities of Z_R outside the critical strip $\operatorname{Re}(s) \in [-h, h]$ have multiplicity $2m_0$.

The differential forms $\phi^*(\Omega_R(\mathcal{P}^+) \wedge \alpha_R^-)$ and $\phi^*(\Omega_R(\mathcal{P}^{+-}) \wedge \alpha_R^+)$ should be regarded as representing a (top-degree) *secondary characteristic class* of the normal bundle of the weak-stable and weak-unstable foliation of $S(X)$, respectively.

There is an equivalent description of $\Omega_R(\mathcal{P}^\pm)$ which emphasizes the analogy of the forms $\Omega_R(\mathcal{P}^\pm)$ with the Pfaffian of the curvature of the Levi-Civita connection of a Riemannian manifold.

In fact, consider the involution J on the tangent bundle $T(G/MA)$ defined by $J|_{\hat{\mathcal{P}}^\pm} = \pm id|_{\hat{\mathcal{P}}^\pm}$ and set $B(X, Y) = \Omega(X, JY)$ for the G -invariant symplectic form Ω on G/MA obtained by reduction of the canonical symplectic form on $T(G/K) \setminus 0$. The leaves of the stable and unstable foliations (of G/MA) are Lagrangian submanifolds with respect to Ω . Then B is an invariant pseudo-Riemannian metric of signature $(d-1, d-1)$. The curvature 2-form ω_D of the corresponding torsion-free pseudo-Riemannian connection D (also considered in [13]) splits as

$$(14) \quad \omega_D = \begin{pmatrix} \omega_D^+ & 0 \\ 0 & \omega_D^- \end{pmatrix}, \quad \omega_D^- = -\omega_D^+$$

according to the G -invariant decomposition $T(G/MA) = \hat{\mathcal{P}}^+ \oplus \hat{\mathcal{P}}^-$. Then the lift (via $G/M \rightarrow G/MA$) of the determinant of $(i/2\pi)\omega_D^\pm$ coincides with the form $\Omega_R(\mathcal{P}^\pm)$ up to an exact basic form.

The differential forms $\Omega_R(\mathcal{P}^\pm) \wedge \alpha_R^\pm$ also can be regarded as the top-degree components of A -equivariantly closed forms (of mixed degree) on $\Gamma \backslash G/M$. Therefore Theorem 1 can be regarded as a *regularized analog* of a localization

formula in equivariant cohomology (see [1]). In particular, it is natural to regard the multiplicity formula (13) as an analog of the Poincaré-Hopf formula for the sum of the indices of a (non-degenerate) vector field.

In the case of a compact Riemannian surface $X = \Gamma \backslash H^2$ equation (4) follows from Theorem 1 by an elementary calculation. In fact, in this case the differential forms

$$(4\pi^2)\phi^*(\Omega_R(\mathcal{P}^+) \wedge \alpha_R^-)$$

represent the Godbillon-Vey class of the weak-stable foliation of the geodesic flow on the sphere bundle $S(X)$ of the surface X . Thus Theorem 1 can be interpreted as the assertion that the Godbillon-Vey invariant of the weak-stable foliation can be calculated from the closed orbits of the geodesic flow. The relation between the Godbillon-Vey invariant of the weak-stable foliation of $S(X)$ and the Euler characteristic of the surface X is a well-known famous result due to Roussarie (see [9]).

By the method of symbolic dynamics it follows that m_0 coincides with the alternating sum of the (finite) dimensions of the generalized eigenspaces of the transfer operators $L_*(0)$ for the eigenvalue 1. Recall that the operators $L_*(0)$ do *not* depend on return times (see [5])! Therefore Theorem 1 also can be regarded as a formula for the integer associated to the flow by forming this *analytical index*. Although the definition of the latter index strongly depends on the choice of a Markov-family of local sections, it is, in fact, independent of the ambiguities involved in the construction of the local sections. Moreover, Theorem 1 shows that the analytical index coincides with the integer defined by integrating a secondary characteristic cohomology class that depends only on the hyperbolic structure of the flow.

Theorem 1 is but a special case of more general formulas relating the multiplicities of the singularities of generalized Selberg zeta-functions at special points to integrals of canonically associated secondary characteristic classes.

Next we combine Theorem 1 with proportionality theory. Let Y^d be the compact dual symmetric space of Y . Y^d is a rank one space, and all geodesics are closed and have the same length. Let Y_{geo}^d be the space of all (oriented) geodesics in Y^d (see [2]).

Theorem 2. *Let X be as in Theorem 1. Then*

$$(15) \quad m_0 = \left(\chi(X) / \chi(Y^d) \right) \chi(Y_{\text{geo}}^d),$$

where χ always denotes Euler characteristic.

Moreover, by a calculation of the Euler characteristics of the compact homogeneous spaces occurring in (15) one obtains

Corollary 1. *Let X be as in Theorem 2. Then*

$$(16) \quad m_0 = (\dim(X)/2)\chi(X).$$

Note that, in contrast to the even-dimensional case, for an odd-dimensional real hyperbolic space we have the following formula.

Theorem 3. *Let $X = \Gamma \backslash H^{2n+1}$ be a compact real hyperbolic space of dimension $2n + 1$. Then the multiplicity of the singularity of Z_R in $s = 0$ is given by*

$$(17) \quad 2 \left((-1)^{n+1} b_{n+1}(X) + \cdots + (-1)^{2n+1} (n+1) b_{2n+1}(X) \right)$$

where $b_p(X)$ is the p th Betti-number of X .

On the proofs. Our proof of Theorem 1 is, unfortunately, much more complicated than the result itself suggests. Thus we can give here only some hints on how the assertion can be proved. More details can be found in [12].

The proof rests on a cohomological trace formula which can be regarded as a common (non-commutative) generalization of the Poisson summation formula and the Lefschetz fixed-point formula. It implies that the zeta-function Z_R is closely related to the alternating product of infinite-dimensional (regularized) characteristic determinants of global Frobenius-operators (canonically determined by the action of the geodesic flow) on the cohomology groups of some differential complexes associated to the invariant foliations of $S(X)$. More precisely, it yields a cohomological formula for the multiplicity m_0 in terms of the Lie-algebra-cohomology of n^\pm with values in the Harish-Chandra modules of the irreducible representations of G in $L^2(\Gamma \backslash G)$.

Now if $\dim(X)$ is even, then G has a compact Cartan subgroup H . The cohomological formula for m_0 turns out to be connected with an analogous cohomological formula obtained by replacing n^\pm -cohomology by Lie-algebra-cohomology with respect to the nilradicals of the (complex) Borel algebras containing the complexified Lie algebra of H . This can be proved by using Osborne's character-formula (see [10]) and suitable patching conditions for characters on neighbouring Cartan subgroups. But the latter number coincides with the analytical index of the (elliptic) deRham complex on the space $\Gamma \backslash G/H$. By working backwards with the corresponding index-form (given by Gauss-Bonnet), one finally ends up with the formula (13).

If the dimension of X is odd, then one can explicate the cohomological formula for m_0 directly by using results from [4].

It is also possible to give more traditional proofs by applying suitable explicit Selberg trace formulas. A proof of Theorem 3 along these lines can be found in [6]. In the even-dimensional case a proof of Corollary 1 resting on Selberg trace formulas is given in [3]. Finally, for quaternionic hyperbolic spaces a representation theoretical proof of Corollary 1 is given in [15].

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