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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1995

A. *Normal families of meromorphic functions*, by Chi-tai Chuang. World Scientific, Singapore, 1993, xi+473 pp., $68.00. ISBN 981-02-1257-7

B. *Normal families*, by Joel L. Schiff. Springer, New York, 1993, ix+236 pp., $39.00. ISBN 0-387-97967-0

Compactness of families of functions is the basis of a most important and deep method of constructing functions. For example, the simplest way to solve (locally) the differential equation

$$y' = f(x, y),$$

with f merely continuous, is to take a subsequential limit of an appropriate sequence of continuous functions.

In 1906 Paul Montel showed that a family of analytic functions on a domain $\Omega \subset \mathbb{C}$ is equicontinuous if it is uniformly bounded on compacta of Ω. This suggested that the theory of compact families would have more coherence and integrity in the context of analytic functions, and developments since Montel confirm this. For example, it is now the basis of the most standard proofs of Riemann's mapping theorem.

Montel defined a normal family as a set of analytic or meromorphic functions $\mathcal{F}$ on an open set Ω such that any sequence $f_n \subset \mathcal{F}$ contains a subsequence g_n converging uniformly on compact subsets of Ω to an analytic (meromorphic) function, possibly constant (including the infinite constant). The Montel theory (and more, as we shall see) is summarized in his monograph [9]. Sometimes one may demand that the limit function itself be in $\mathcal{F}$; in this case $\mathcal{F}$ is called *closed*, and this possibility is considered in (A).

About a decade after Montel's first findings, Fatou and Julia based their theory of iteration on his theory. It allowed sophisticated techniques to be transferred to an area which in the past had largely been devoted to ad hoc analyses of functional equations. A fine account of this is given in [1]. In view of the recent rebirth of activity in complex iteration, it is appropriate that modern treatments of normal family theory also appear. The two books under review do this, at least in the setting of one complex variable. Both presume only a standard first course, and (B) is especially conveniently arranged. They have quite different emphases but have the Montel and some post-Montel work in common.

I think the crowning achievement of Montel's work is his theorem that a family $\mathcal{F}$ of meromorphic functions which omit three fixed values is normal. (Schiff calls this the FNT: fundamental normality criterion.) This gave a significant interpretation of Picard's theorem and was the single most important result used by Fatou and Julia. Applications of the FNT to the theorems of Schottky, Landau, and Bloch and existence of Julia directions occur in both books.

In one complex variable, a few key directions were pursued post-Montel. The two above-cited results by Montel suggested that properties P which reduce an entire/meromorphic function in the plane to a constant might render a family $\mathcal{F}$ of functions normal. There is a weak form of this principle in Bloch [4, p. 84], as cited in (A), page 75; it is explicitly stated (and credited to no one) on page 250 of [7] and is the basis of several problems posed in [6].

This so-called Bloch principle has led to an exhaustive catalogue of normal-family criteria, well covered in (B) and less exhaustively in (A); (B), page 106 also considers the converse principle. In addition, (B), §4.2 discusses examples which show that this principle can fail in either direction. (The most elementary counterexample, shown to me by Eremenko, is: let P be the property that the spherical derivative $f^{\#}(z) = |f'(z)|(1 + |f(z)|^2)^{-1} < M$ for some fixed M. It follows at once from the elementary Marty criterion that a family $\mathcal{F}$ which satisfies P is normal. But meromorphic functions which satisfy P are legion: e^z, $\sin z$, $\cos z$, $\wp(z)$, ...). Nonetheless, it remains a useful metaphor in suggesting normality criteria. One splendid example of its success is due to Gu [5]:

Theorem. *Let D be a domain, $a, b \neq 0$ two complex numbers and $k \geq 1$ an integer. Let f be a family of meromorphic functions in Ω such that the equations*

$$f(z) = a, \; f^{(k)}(z) = b$$

have no solutions D. Then $\mathcal{F}$ is normal.

Indeed the corresponding theorem for functions in the plane (by Hayman) stood alone for twenty years before Gu's theorem gave it a normal family mate. Of special note is that the hypotheses involve *two* values rather than *three*, as in Picard-Montel. These normal-family analogues are always more complicated to establish than results for a single function; this occurs even in Montel's proof of the FNT.

For many years, the standard way to connect criteria in these two contexts was by Nevanlinna's theory of meromorphic functions, based on the Poisson-Jensen formula and elementary, if sophisticated, lemmas about the growth of increasing real functions. When it works, the parallel theories can synchronize in a compelling manner, but to force the sharpest normal family analogues of Picard-type criteria can often lead to a rococo labyrinth of cases and subcases and an awkward handling of certain "initial values terms", which have to be controlled for each $f \in \mathcal{F}$. Zalcman [15] brought a new ingredient into the theory by writing down an analytic condition implied by nonnormality. A family $\mathcal{F}$ of functions in Ω is *not* normal if and only if there exists a compact subset $\Omega' \subset \Omega$ and $z_n \in \Omega'$, $f_n \in \mathcal{F}$, $\rho_n \to 0$ such that

$$f_n(z_n + \rho_n \zeta) \to g(\zeta) \quad (n \to \infty) \tag{1}$$

uniformly on compacta of $\mathbb{C}$ with g a nonconstant function entire/meromorphic in the plane. This has been applied by Schwick [13] to obtain normal family criteria, and these criteria in turn have been used most recently by Bergweiler and Fuchs [3] in their study of real zeros of derivatives of entire functions of infinite order. It can often be used to simplify many of the complications which arise when applying Nevanlinna theory to these questions; this was first done by Oshkin [10].

More significantly, Pang [11, 12] has recently given modifications of both Zalcman's nonnormality criterion and the Bloch principle formalism which have been able to handle normality conditions which involve derivatives as well as the new result that if $\mathcal{F}$ is a family of meromorphic functions with $f'f^2 \neq 1$, $f \in \mathcal{F}$, then $\mathcal{F}$ is normal. (That $f'f^n \neq 1$, with fixed $n > 2$, is a normality condition was established earlier by several authors and is discussed both in (B) and, more expansively, in the more recent [16].) These results are most striking, since now we have conclusions which follow from just *one* omitted value. Pang proves that if $\mathcal{F}$ is not normal and k, $-1 < k < 1$, is given, there exist z_n, f_n, and ρ_n as above such that

$$\rho_n^{-k} f_n(z_n + \rho_n \zeta) \to g(\zeta); \tag{1'}$$

the flexibility allowed by introducing this weighted form permits study of criteria involving derivatives, lets many existing proofs be simplified, and certainly will have further impact. Since [11] and [12] are not widely available, it is regrettable that (B) does not give a proof of Pang's results.

One of the remaining open problems of interest in this subject is whether the property $f'f \neq 1$ is a normal-family criterion for meromorphic families. Pang's work shows that an affirmative answer will follow from the (likely) simpler question of whether a meromorphic function f with $ff' \neq 1$ must be constant. (If f is of finite order, this has been recently shown by Bergweiler and Eremenko [2], as a consequence of analysis of the possible singularities of the inverse function, but the general case remains open.)

(*Added in Proof*: Both a revised version of [2] and a preprint, *On the value distribution of $f^n f'$* by Chen Huaihui and Fang Mingliang, have settled this question in the affirmative.)

Once the basic material is covered, the books' emphases diverge. Developments through the "Bloch principle" occupy the first four chapters of (B), to which are added about fifty pages of "general applications" and a short appendix on quasi-normal families: families $\mathcal{F}$ such that each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{g_n\}$ which converges normally on compact subsets of $D \backslash Q$, where Q is a finite set, which may depend on the $\{g_n\}$. The set of polynomials of uniformly bounded degree is a standard and natural example of such a family. When the cardinality of Q is at most n, we say $\mathcal{F}$ is quasi-normal of order n.

In contrast, quasi-normal families, extensions, and applications occupy about two-thirds of (A). The author, along with K.-L. Hiong (1893–1969), may be considered the founder of function theory in China. His connections with French mathematicians is of about sixty years' standing, dating from Hadamard's 1935–36 visit to Tsinghua University, followed by his study in France, where G. Valiron was his teacher. From Chuang's point of view, normal and quasi-normal families are Q_0 and Q_1 families, and the last two chapters of (A) introduce Q_m families for any natural number m, together with applications, many of which are due to him. Work of Hiong, Chuang, L. Yang, G.-H. Zhang, Gu, X. Li, X. Pang, and others in the theory of normal families has continued this area through the decades and is well documented in both books; these authors and others have also obtained significant results in other areas of function theory, in particular value-distribution theory.

Applications play a more significant role in (B). The Fatou-Julia theory of rational iteration is sketched (modulo details concerning the number of indifferent periodic points) through Julia's theorem that the Julia set is the closure of repulsive fixed points. (Montel's own text [9] does include such details.) (A) assumes the reader's familiarity with Nevanlinna theory, while (B) gives a substantial outline.

Of course, one can iterate entire functions, and (B) presents I. N. Baker's theorem of 1968 that Julia's theorem holds in this setting. Baker's theorem is based on an insightful application of Ahlfors's theory of covering surfaces; so to make his account more self-contained, Schiff includes a short sketch of the Ahlfors theory. Quite recently Schwick [14] gave a very short proof of Baker's theorem independent of the Ahlfors theory (however, the extension of Baker's theorem to semigroups generated by families of functions, due to A. Hinkkanen and G. Martin, still requires the full power of Baker's analysis, and [14] itself depends essentially on nontrivial Nevanlinna theory). Further appendices discuss harmonic functions, extremal problems, and normal families of Möbius transformations, in particular discontinuous groups.

Schiff's book is smaller than Chuang's, and it is splendidly organized. There is an extensive bibliography, designed to make it routine to locate the sources of the main theorems. There are also many notes quoting the more refined forms of the theory. Even a motivated undergraduate should be able to handle much here; the calculus of sequences, subsequences, and the various metrics are displayed in all detail—so much so that a more advanced reader may skip large chunks of material. (A) quotes few original sources, and it is sometimes harder, for example, to locate a definition used repeatedly through a chapter. (A) has a few minor typos; it seems that (B) is essentially error free.

In one respect, both (A) and (B) are a step backward from Montel's text [9]; I think Montel's horizons are somewhat broader, including much on several complex variables. Montel's Chapter X is devoted to n-tuples of analytic functions such that $\sum_1^n f_j = 0$; some issues raised therein are not yet settled; see the last chapter of [8]. Normal families are used in several variables, often to analyze boundary behavior of biholomorphic mappings, in deducing hyperbolicity criteria or in developing analogues of Bloch's theorem. However, there seems to be no counterpart of the one-variable study of derivatives or combinations of derivatives, as in Gu's theorem. Montel also has a detailed study of the boundary behavior of the Riemann map (in one complete variable), algebroid functions, and the uniformization of algebraic curves.

Acknowledgment

I thank W. Bergweiler, A. Eremenko, L. Yang, S. Krantz, and M. Ru for helpful comments and L. Zalcman for a preprint of [16].

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1995

0273-0979/95 $1.00 + $.25 per page

Convex bodies: The Brunn-Minkowski theory, by Rolf Schneider. Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge Univ. Press, Cambridge, 1993, xiii + 490 pp., $89.95. ISBN 0-521-35220-7

Let me first describe two (main) classical results which form the backbone of the book.

Brunn-Minkowski inequality: *Let A and B be convex bodies in $\mathbb{R}^n$ and Minkowski addition $A+B=\{z=x+y \mid x\in A\ ,\ y\in B\}$. Then*

$$(*)\qquad \left[\mathrm{Vol}(A+B)\right]^{1/n} \geq (\mathrm{Vol}\,A)^{1/n} + (\mathrm{Vol}\,B)^{1/n}\ .$$