BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1995

Topology of gauge fields and condensed matter, by Michael Monastyrsky (translated from Russian by Oleg Efimov). Plenum Press, New York, 1993, 372 pp., $95.00. ISBN 0-306-44336-8

This book is a contribution to the dialogue between mathematical physics, geometry, and algebraic topology. The interaction between these subjects has been such a dominant feature of research developments in the past few years that it seems scarcely necessary to recite a list of examples: in fact, on the mathematical side it is quite hard to think of active areas in geometry and topology which have not been noticeably influenced by insights from physics—where by "physics" we mean particularly quantum field theory—and on the other hand the geometrisation of fundamental physical concepts is a profound and pervasive development. There is therefore a very good a priori case for a book such as this one, which gives an account of some modern geometry and physics and particularly the connections between them.

The first part of the book, comprising two chapters and 170 pages, is purely mathematical and takes the reader on a rapid trip through the elements of manifold theory—the definition via an atlas of local charts, vector fields, differential forms, etc.; differential geometry, particularly the fundamentals of Lie groups, fibre bundles and connections, and algebraic topology; and homotopy and homology groups, with an emphasis on de Rham cohomology. Much of this material is brought together at the end of this section with a discussion of the Chern-Weil theory of characteristic classes. The bulk of this is standard material, usually covered in beginning graduate courses for mathematicians, and is dealt with in a number of well-established texts. The distinctive features of the book under review are that it brings the material together in one place and that it gives a condensed treatment, omitting nearly all proofs. This may make it easier to get an overall picture, which the author has tried to make more accessible to his less-mathematical readership with a discussion based more on examples and less on the conventional mathematician's pattern of abstract definitions and proofs. A great deal of information is packed in, going in some directions beyond the standard syllabus envisaged above; for example, there is a discussion of the homotopy groups of Lie groups and Bott periodicity and a detailed treatment of the Whitehead product on homotopy groups. On the other hand, the exposition is spoiled by mistakes of various sorts. There are quite a number of typographical errors which, although minor in themselves, could cause substantial confusion (for example, in the statement of the Bott periodicity theorem for the orthogonal groups). There are also some mathematical mistakes, confused arguments and definitions; in addition, the organisation of the details of the material could be improved in some places: for example, the fundamental idea of a tangent vector field as a differential operator on functions, used in the definition of the Lie bracket, is not explained clearly. Perhaps the main criticism of this part of the book is that it is hard to get a sense of the relative difficulty and importance of the various ideas that are presented. Difficult theorems, such as the Hodge theorem and the topological invariance of homology (defined via cellular decomposition), are mentioned, but the uninformed reader may not perceive the difference in depth of these results from, for example, the Fundamental Lemma of Riemannian Geometry on the existence and uniqueness of the Levi-Civita connection. Of course this is to some extent an inevitable problem with a broad survey such as this, in which the reader is referred elsewhere for nearly all proofs (the author does give many references to more comprehensive works), and one should not expect too much. In sum, this first part of the book gives an overview of a lot of mathematical material which forms a basis for the discussion of physical problems in the second half and which will make an interesting complement to a study of more detailed texts but which will probably be hard to understand in isolation by someone who has not met many of the ideas before.

We turn now to the second, and more distinctive, half of the book: the discussion of applications in gauge field theories and the physics of condensed matter. The starting point here is now quite well known: expressing the Maxwell equations for an electromagnetic field over Lorentz space as the Euler-Lagrange equations for a Lagrangian defined on the connections on a $U(1)$ bundle, where the electromagnetic potential becomes the connection and the field tensor its curvature. The freedom of choice of "gauge" for the potential is a fundamental fact which stems, in the geometrical picture, from the lack of a preferred trivialisation of the bundle. Once these ideas are in place it is a small step to the more general Yang-Mills theories, in which the group $U(1)$ is replaced by any Lie group. In addition to the pure connection and its curvature, one also considers other fields, which are sections of associated bundles. One of the author's main goals is to explain the "Higgs mechanism" for the creation of particle masses. In the simplest case of a field ψ which is a complex valued function the Lagrangian

$$\int |\nabla\psi|^2 + m|\psi|^2$$

leads to a linear Euler-Lagrange equation $\Delta\psi + m\psi = 0$, and m is interpreted physically as the *mass* of the particle described by ψ. More generally, in a situation where we have a ground state Ψ_0 minimising an action functional S, one can consider small variations $\Psi_0 + \psi$ about the ground state and take the quadratic terms in the action functional which will typically have the form

$$S(\Psi_0 + \psi) = S(\Psi_0) + \int |\nabla\psi|^2 + m|\psi|^2 + O(\psi^3).$$

Then the coefficient m is interpreted as the mass of the particle associated to the ground state. In the case of vector-valued wave functions, one diagonalises the corresponding quadratic form to get a number of different particles, with masses given by the eigenvalues. Now in the case of pure Yang-Mills we have to regard the connection A as the variable, with ground state given by the flat connection $A = 0$. Then we can apply the procedure above to the Yang-Mills Lagrangian

$$S(A) = \int |F|^2 = \int \left| dA + \frac{1}{2}[A, A] \right|^2 ,$$

but taking quadratic terms in A we get

$$S(A) = \int |dA|^2 + O(A^3).$$

The only quadratic term involves derivatives of A, which means that the corresponding particle would have zero mass: a "massless vector boson", and these are undesirable on physical grounds. This phenomenon is an inevitable consequence of gauge invariance, since it is not possible to write down any gauge-invariant Lagrangian containing quadratic terms in A rather than its derivatives.

Suppose, on the other hand, that the gauge field is absent but that we consider a more general Lagrangian $|\nabla \psi|^2 + U(\psi)$, where ψ takes values in a vector space V and U is invariant under a symmetry group G: for example, $V = \mathbf{C}$, $G = S^1$ and $U(z) = |z|^4 - |z|^2$. Then a ground state is given by any constant function Ψ_0, whose value is a minimum of the function U on V. Now this ground state will not in general be unique but a whole orbit of G in V: in other words, the choice of a ground state breaks the symmetry present in the underlying set up. This symmetry breaking again leads to undesirable massless particles: the Hessian of the function U is forced to vanish in directions tangent to the G-orbit of Ψ_0.

Surprisingly, these two objectionable phenomena disappear when we put the two situations—a gauge field and spontaneous symmetry breaking—together to consider a Lagrangian involving a connection and an auxiliary field. This is because the choice of a ground state also gives a trivialisation of the relevant bundle (or at least a preferred class of trivialisations) which absorbs the gauge freedom. In physical language, the interaction of the gauge field and the auxiliary field endows the associated particles with nonzero masses.

This discussion of gauge invariance, symmetry breaking, and the Higgs mechanism makes a fine illustration of the application of the differential geometric background material. One of the other main topics is the existence of "monopole-like" solutions, which illustrate the role of topological ideas. Here one considers again a Lagrangian involving a connection and auxiliary field over $\mathbf{R}^3$. The minimum of the potential function U is a G-orbit $G/H \subset V$, and assuming suitable finiteness conditions at infinity in $\mathbf{R}^3$, the asymptotic values of the field give a map from the 2-sphere to the orbit G/H. One is interested in situations where this map is homotopically nontrivial. The homotopy class then appears as a kind of topological charge of the system, which gives a topological mechanism for the creation of solitons: solutions which behave at large distances from the origin like magnetic monopoles. Thus one is interested in homogeneous spaces G/H with nontrivial homotopy groups $\pi_2(G/H)$, and the book explains how to analyse these, using the techniques developed in the first part (principally the long exact sequence of the fibration $H \to G \to G/H$).

The last part of the book treats some topological aspects of the physics of condensed matter. The general mathematical setup here is that one has a material which can be described by a map ϕ from a region in 3-space to a manifold M of "internal states": in the simplest example of a "nematic liquid crystal" the space M is the projective space $\mathbf{RP}^2$ of directions in $\mathbf{R}^3$ and the value $\psi(x)$ represents the spatial orientation of a liquid crystal at a point x. One is interested in the case when there are topological singularities, or defects, in the structure, i.e., the domain Ω is the complement of a collection of curves and points in $\mathbf{R}^3$ and the map ϕ cannot be extended continuously, even up to homotopy, over these subsets. Thus for each point in the singular set there is an obstruction in $\pi_2(M)$ to this extension, given by restricting to a small sphere about the point, and for each loop in the singular set an obstruction in $\pi_1(M)$, given by restriction to a small linking circle (but one needs to take care with base points, an issue which becomes important in the theory). This leads to quite an involved discussion combining knot theory with the homotopy theory of the internal space state M.

In conclusion, this book fulfills its objective of illustrating the interplay between geometric and topological techniques and physical questions and contains a great deal of interesting material, sometimes presented in a rather obscure fashion. It makes slightly frustrating reading, at least for the reviewer, not so much because of any mathematical shortcomings, but because the author stops short of giving a clear and full account of the background in physics. On the one hand, the reader will probably have to be reasonably familiar with some of the background and notation to make much headway; on the other, some fundamental aspects of the theory are hardly touched on; for example, Feynman Integrals appear very briefly in motivating the study of instanton solutions, and one would like to have been told more about this part of the story—but perhaps this is asking for too much. The book is a worthwhile addition to the literature, but one is left feeling that there is an important gap in this literature which remains to be filled in presenting these extremely important concepts from modern physics to mathematicians.

S. K. DONALDSON
UNIVERSITY OF OXFORD
E-mail address: donaldson@vax.ox.ac.uk

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1995

0273-0979/95 \$1.00 + \$.25 per page

The Riemann zeta function, by A. A. Karatsuba and S. M. Voronin. de Gruyter, Berlin and Hawthorne, New York, 1991, xii + 396 pp., \$112.00. ISBN 3-11-013170-6

The Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\operatorname{Re}(s) > 1$. It has a meromorphic continuation to $\mathbb{C}$ with the only pole being at $s = 1$. Moreover it satisfies a functional equation relating s and $1-s$. Its relevance to prime numbers lies in the relation $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$. The central problem concerning $\zeta(s)$ is *Riemann's Hypothesis* (RH) that the zeros of $\zeta(s)$ in $0 <$