[Mun] Munkres, J., *Topological results in combinatorics*, Michigan Math. J. **31** (1984), 113-128.

[PS1] Peskine, C. and L. Szpiro, *Dimension projective finie et cohomologie locale*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 323–395.

[PS2] ———, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 1421–1424.

[Re] Reisner, G., *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. **21** (1976), 30–49.

[Ro1] Roberts, P., *Le théorème d'intersection*, C. R. Acad. Sc. Paris Sér. I Math. **304** (1987), 177–180.

[Ro2] ———, *Intersection theorems*, Commutative Algebra, Math. Sci. Res. Inst. Publ., vol. 15, Springer-Verlag, New York, Berlin, and Heidelberg, 1989, pp. 417–436.

[S] Serre J.-P., *Algèbre locale. Multiplicités*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin, Heidelberg, and New York, 1965.

[St1] Stanley, R., *Cohen-Macaulay rings and constructible polytopes*, Bull. Amer. Math. Soc. **81** (1975), 135-142.

[St2] ———, *The upper bound conjecture and Cohen-Macaulay rings*, Stud. Appl. Math. **54** (1975), 133–142.

[St3] ———, *The number of faces of a simplicial convex polytope*, Adv. Math. **35** (1980), 236-238.

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Multivalued differential equations, by Klaus Deimling. de Gruyter, Berlin and New York, 1992, 260 pp., $59.00. ISBN 3-11-0132125

Although some papers on multivalued differential equations, or differential inclusions, appeared in the literature before the middle of the century, the subject began to interest mathematicians seriously in the 1960s. Several motivations concurred. On one hand there was the interest in control theory and in optimal control. In an ordinary differential equation such as $x'(t) = f(t, x(t))$, an additional parameter u (the control) is introduced, and one considers the controlled system $x' = f(t, x, u)$, $u \in U$, where U is the set of admissible controls. Hence, at a given time t and a given state $x(t)$, several velocities are possible: $x'(t)$ has to belong to the set $\{f(t, x(t), u) : u \in U\}$. So what really counts for the dynamics is this set, which can be described by a set-valued map F: from $\mathbb{R} \times \mathbb{R}^n$ to the nonempty subsets of $\mathbb{R}^n$, and the controlled differential equation becomes an inclusion, i.e.,

$$x'(t) \in F(t, x(t)).$$

The above has been an important motivation for studying differential inclusions and has led to the simplification of some proofs on the existence of optimal controls [8]. At about the same time, interest arose in nondifferentiable functions. A continuous but not differentiable convex function admits subdifferentials. Minima of a functional of the calculus of variations of the kind

$$\int_I g(t, x(t), x'(t))\, dt$$

when g is only assumed to be convex will not satisfy a classical Euler-Lagrange equation, since g is not differentiable. However, primarily through the work of R. T. Rockafellar and his school, it was shown that it satisfies a suitably defined system of differential inclusions. In general, whenever we have to face the lack of smoothness or of uniqueness of a choice defining derivatives, we have to deal with a differential inclusion.

As strange as it may seem, the problem of proving the existence of a solution to the standard Cauchy problem for a differential inclusion is far more difficult than the corresponding problem for a differential equation. What is missing, in the case of an inclusion, is the direct link between a function $x(t)$ and its derivative $x'(t)$ expressed by the equation $x'(t) = f(t, x(t))$. For a converging sequence of solutions (x_n) the above equation implies the convergence of the derivatives (x'_n), and essentially the same conclusion is true for a converging sequence of approximate solutions. When the right-hand side becomes a set, the convergence of (x_n) has no influence on the convergence of (x'_n) and other methods of proof have to be devised.

There are different kinds of differential inclusions, essentially as there are different kinds of partial differential equations. Of the multifunction appearing at the right-hand side we can describe different kinds of continuity: there are upper and lower semicontinuity, as there are Hausdorff continuity and Lipschitzeanity. Or we can describe the geometric or topological properties of the images of the map F: these images can be closed sets or may be convex sets, compact or unbounded, or can have nonempty interior. In general, when the map F is compact and convex valued, the standard approach (as used in the calculus of variations since the beginning of the century) of passing to the weak L^1 convergence of the derivatives of approximate solutions and exploiting the convexity of the images of F leads easily to the proof of the existence of solutions. But when convexity of the images of the right-hand side is not assumed, the difficulties begin. And this at least should be said about this subject: in the existence theory we face the challenge posed by the lack of convexity in its simplest and most direct way. For this reason in the last thirty years differential inclusions have been a formidable gymnasium for the creation of ideas that would help to overcome this difficulty: the methods devised range from the strong compactness construction of Filippov [9] (a theorem that still, after so many years, I consider very clever), to the continuous selection method [1], to the introduction of Lyapunov's Theorem on the range of measures [11], to the Baire Category theorem used as a tool to prove existence of solutions [6], to the existence of directionally continuous selections [4]. The results obtained were, sometimes, very different from what one would expect. Consider the following example, due to De Blasi and Pianigiani [7]. In an infinite-dimensional Hilbert space X there exists a continuous set-valued map F, from $\mathbb{R} \times \mathbb{R}^n$ to the closed bounded subsets of X such that the set of all continuous selections is countable (i.e., a sequence of continuous functions $(f_i, i \in \mathbb{N})$ describes all the continuous selections of F) and none of the Cauchy problems,

$$x'(t) = f_i(t, x(t)), \qquad x(t_0) = x_0, \qquad i \in \mathbb{N},$$

admits a solution, while the problem

$$x'(t) \in F(t, x(t)), \qquad x(t_0) = x_0$$

admits a solution.

The books that have appeared so far on the subject of differential inclusions [2, 3, 9, 12] all devote a substantial amount of pages to the topic of existence theorems, and the book by Deimling is no exception. It covers many aspects of the existence theory, both in the upper semicontinuous and the lower semicontinuous cases and both in finite- and infinite-dimensional spaces. One is impressed by the remarkable effort to discuss or, at least, to mention in the "Remarks" at the end of each chapter, a large amount of material, larger than the size of the book would lead one to expect. Eventually the case presented more completely is the upper semicontinuous case for maps whose values are compact and convex sets. This is the case where differential inclusions most resemble ordinary differential equations. Actually one can show, by a simple argument [4], that they are indeed limits of ordinary differential equations. So no really new phenomena on the properties of the solution sets should occur. This case is considered "too easy" by active mathematicians; but it must be said that when a differential inclusion actually occurs in practice, in general the map appearing at the right-hand side is upper semicontinuous. It is true, though, that it need not be convex valued: upper semicontinuity without convexity is possibly the most interesting case, still an active subject of research. One might wish that the book by Deimling would contain some material on the properties of the solution sets to non-convex-valued differential inclusion, a topic that in recent years has reached very interesting and complete results, at least in the Lipschitzean case.

Besides the chapters on the existence theory and on the properties of the solution sets, Chapter 5 presents applications and contains different topics as boundary value problems, existence of periodic solutions, and stability and asymptotic behavior of the solutions. Essentially, in these paragraphs, techniques are presented to extend the results known in the case of ordinary differential equations to the case of upper semicontinuous differential inclusions with compact convex right-hand side. Deimling collects and simplifies the presentation of a good amount of material; it is a useful job that had to be done, especially because the techniques involved are not always exciting.

The book offers problems to be solved; the style of the presentation is bright, sometimes a little personal, in particular in the quotations appearing in the "Remarks" paragraphs—all in all, a welcome addition to the literature that often presents material at the edge of current research.

References

1. H. Antosiewicz and A. Cellina, *Continuous selections and differential relations*, J. Differential Equations **19** (1975), 386–398.

2. J. P. Aubin, *Viability theory*, Birkhäuser, Basel and Boston, MA, 1992.

3. J. P. Aubin and A. Cellina, *Differential inclusions*, Springer, New York, 1984.

4. A. Bressan, *Directionally continuous selections and differential inclusions*, Funkcial. Ekvac. **31** (1988), 459–470.

5. A. Cellina, *Multivalued differential equations and ordinary differential equations*, SIAM J. Appl. Math. **18** (1970), 553–558.

6. F. De Blasi and G. Pianigiani, *A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces*, Funkcial Ekvac. **25** (1982), 153–162.

7. ———, *The Baire category method in existence problems for a class of multivalued differential equations with nonconvex right hand side*, Funkcial. Ekvac. **28** (1985), 139–156.

8. A. F. Filippov, *On certain questions in the theory of optimal control*, SIAM J. Control Optim. **1** (1962), 76–84.

9. ———, *On the existence of solutions of multivalued differential equations*, Mat. Zametki **10** (1971), 307–313.

10. ———, *Differential equations with discontinuous right-hand sides*, Kluwer, Dordrecht, 1988.

11. A. Fryskowski, *Continuous selections for a class of nonconvex multivalued maps*, Studia Math. **76** (1983), 163–174.

12. M. Kisielewicz, *Differential inclusions and optimal control*, PWN, Warsaw, 1991.

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Operator-limit distributions in probability theory, by Zbigniew J. Jurek and J. David Mason. Probability and Math Statistics, Wiley, New York, 1993, xii + 292 pp., $79.95. ISBN 0-471-58595-5

The central limit theorem is one of the cornerstones of probability theory. The importance of the central limit theorem is that it allows the probabilistic behavior of the sum of a large number of independent random variables to be approximated by the probabilistic behavior of a single random variable which is often simpler (in some sense) than the summands themselves. This is especially true in the case of the familiar statistician's central limit theorem. This result states that if $\{X_j\}$ is a sequence of independent and identically distributed random variables with finite variance, then for large n the distribution of the sum $\frac{1}{\sqrt{n}}\sum_{j=1}^n(X_j - E[X_j])$ is approximately a normal distribution.

There are at least two useful directions for generalization of the statistician's central limit theorem. First, the assumption that the $\{X_j\}$ are identically distributed and have finite variance can be relaxed. Second, the random variables $\{X_j\}$ can be replaced by random vectors. Both of these directions are natural from a statistical point of view, since each observation may record several measurements on a single experimental unit.

Exploration of both of these directions leads to the following problems. Given a sequence $\{X_j\}$ of independent random vectors, find conditions under which there are linear operators $\{A_n\}$ and vectors $\{a_n\}$ so that

$$A_n \sum_{j=1}^{n} X_j + a_n$$

converges in distribution to some random vector Y. Also, characterize the random vectors Y which can arise in this way, identify the relationship between

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