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*Convex bodies: The Brunn-Minkowski theory*, by Rolf Schneider. Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge Univ. Press, Cambridge, 1993, xiii + 490 pp., $89.95. ISBN 0-521-35220-7

Let me first describe two (main) classical results which form the backbone of the book.

Brunn-Minkowski inequality: *Let $A$ and $B$ be convex bodies in $\mathbb{R}^n$ and Minkowski addition $A + B = \{z = x + y \mid x \in A\ ,\ y \in B\}$. Then*

$$[\mathrm{Vol}(A+B)]^{1/n} \geq (\mathrm{Vol}\,A)^{1/n} + (\mathrm{Vol}\,B)^{1/n}\ . \tag{$*$}$$

There is also a complete description of equality cases. (See Chapter 6 of Schneider's book.)

(Brunn [Br1], [Br2] in fact proved this inequality in 1887 for the 3-dimensional case; however, even the 2-dimensional case is non-trivial.) Brunn's main and most beautiful observation was that Steiner symmetrization preserves convexity; as a consequence, the Brunn-concavity principle follows: *Let* $K$ *be a convex body in* $\mathbb{R}^n$, $E$ *a* $k$*-dimensional subspace and* $F = E^{\perp}$ *the* $(n-k)$*-dimensional subspace orthogonal to* $E$. *For* $x \in F$, *let* $\varphi(x) = (\mathrm{Vol}_k[K \cap (x+E)])^{1/k}$. *Then* $\varphi(x)$ *is a concave function on its support.*

To realize the non-triviality of this statement, consider the "trivial" consequence:

$$\textit{if } K = -K \textit{ (centrally symmetric), then } \max \varphi(x) = \varphi(0).$$

(Try to find an independent proof of the statement.)

The triangle inequality (which defines convexity) corresponds then to the case $k = 1$. However, as we see, convexity has many more "hidden" structures.

Minkowski was the first to realize the importance of "$n$-dimensional generalizations" of the whole subject of convexity (developed originally on 2- and 3-dimensional spaces) and applied it for Analytical Number Theory [Min2]. He wrote, I believe, inequality $(*)$ in the above form and analyzed the equality cases. Only in 1935 did L.A. Lusternik [Lus] extend inequality $(*)$ to any (measurable) sets.

The next step done by Minkowski was the introduction of "Minkowski addition", and he proved the following theorem [Min1]:

*The volume of the linear family,* $K = \lambda_1 K_1 + \cdots + \lambda_r K_r$, $K_i$ *are convex sets in* $\mathbb{R}^n$, $\lambda_i \geq 0$ *is an* $n$*th degree homogeneous polynomial in* $\lambda_1, \ldots, \lambda_r$. *That is,*

$$(**) \qquad \mathrm{Vol}(K) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n}),$$

*where the coefficients* $V(K_{i_1}, \ldots, K_{i_n})$ *are chosen to be invariant under permutations of their arguments. The coefficient* $V(K_1, \ldots, K_n)$ *is called the* mixed volume *of the convex sets* $K_1, \ldots, K_n$.

This brought to light the notion of mixed volumes, and a whole new beautiful world of the study of different inequalities between different mixed volumes was opened (see Chapter 5 of the book under review).

We call this now "geometric inequalities", but it includes many new directions of thinking besides extremely deep inequalities discovered by Alexandrov (1936) and Fenchel (1936) (see Chapter 6 of the book).

These last series of inequalities by Alexandrov and Fenchel still hold a mystique and, from my point of view, are not completely understood. They are parallel to Hodge theory (see [BZ, §27]) also from 1936—but this became clear only forty years later—and bring deep connections of convexity with algebraic geometry (through the results of Kuchnirenko-Bernstein-Khovanskii-Tissier: see [BZ]). However, Minkowski at the beginning of the century, already understood some of these inequalities (the so-called "Minkowski Inequality").

The simplest case of $(**)$ in $\dim 2$ for $r = 2$ and for $K_2 = B$ (the euclidean 2-dimensional disc) was discovered by Steiner back in 1840. At the same time Cauchy found the "Cauchy formula" for the surface area of a convex 3-dimensional body $K$ as the spherical average of 2-dimensional volumes of orthogonal projections of $K$ onto hyperplanes. This is trivially generalized on any dimension and is an important first step in proving the $n$-dimensional Steiner formula. It is a very curious historical coincidence that also in the 1840s Minding considered the problem of finding a formula for a number $N$ of solutions of two polynomial equations ($P_1 = 0$ and $P_2 = 0$, say) of two variables (today we would say polynomials in "general position"). As we know today (through Kushnirenko-Bernstein theory) the mixed volume $V(K_1, K_2)$ of Newton's polyhedrons $K_1$ and $K_2$ (in $\mathbb{R}^2$) of these polynomials is responsible for the number $N$. And this mixed volume $V(K_1, K_2)$ is the only non-trivial coefficient in the decomposition

$$\text{Area}(K_1 + \lambda K_2) = \text{Area}\, K_1 + 2\lambda V(K_1, K_2) + \lambda^2\, \text{Area}\, K_2 .$$

However, Steiner proved this formula only in the case $K_2 = B$ (the euclidean disc), which was a most interesting case in geometry but which was, on the contrary, not an interesting case for Minding because $B$ is never the Newton polyhedron of any polynomial! So, the connection was not observed until the 1970s. The euclidean ball was the only interesting case for Steiner because the isoperimetric inequality was the main target.

So, mixed volumes, $n$-dimensional convexity theory, and the start of geometric inequalities waited fifty more years when Minkowski's genius connected it with number theory.

There is in fact another very curious coincidence regarding Alexandrov-Fenchel inequalities from 1936 and Hodge theory from the same year: it took forty years to understand that they describe very similar mathematical structures.

I see three periods in the development of what one may call "classical quantitative" convexity theory and what Rolf Schneider rightly calls "The Brunn-Minkowski Theory". The first was in the 1840s through Cauchy's and Steiner's works; then about fifty years later Brunn-Minkowski's inequality was discovered, and Minkowski shaped our view of convexity theory of the whole of the twentieth century. (Blaschke, Caratheodory, Bonnesen, and many, many others added new facets to this mathematical gem.)

However, again about fifty years later, in 1936, inequalities proved by Alexandrov and Fenchel marked a whole new period of very deep finite dimensional convexity theory through the study of (exact) geometric inequalities of many different types, description of extremal cases, approximation theory, and so on. Of course, the works of Busemann, Hadwiger, Santaló, Rogers, and many other experts in convexity discovered different and unusual turns in this beautiful lasting theory. The author of the monograph, Professor Rolf Schneider, is one of the distinguished experts in the field, whose results have shaped the interest and direction of the study of many other experts over the last three decades. This monograph gives a very detailed account of these three periods of the theory and all the necessary background.

It took another fifty years after Alexandrov-Fenchel's work, in the middle of the 1980s, for the next step in the development of quantitative convexity theory to crystallize.

We now consider the geometric problem from a functional analytic point of view. Consequently, typical for geometry, "isometric" problems and views are substituted by "isomorphic" ones. This became possible with the *asymptotic* approach (with respect to dimension increasing to infinity) to the study of high dimensional convex bodies.

So another major subject of the study was added to convexity: asymptotic properties of some classes of convex sets related to increasing dimension. In particular, the meaning of geometric inequalities was extended. *Isomorphic* geometric inequalities which involved exact dependence of constants on dimension naturally accompany the asymptotic theory of convex bodies.

Asymptotic theory emerged earlier. From the start of the 1970s the needs of geometric functional analysis led to a deep investigation of the linear structure of finite dimensional normed spaces. The culmination of this study was an understanding of the fact that subspaces (and quotient spaces) of proportional dimension behave very predictably. This was the bridge between the problems of functional analysis and the global asymptotic properties of convex sets.

This last direction was just touched on in Schneider's book through the "Notes" to the sections, and I recommend those interested in this stage of convexity theory to consult the books [MSch], [P] and [T-J].

In the not-so-distant past, there was a notable lack of literature on convexity in English. Although a number of important, and by now classical, books and surveys on the subject have been written in German and Russian, none were translated into English. Only as recently as 1987 was the old "classic" by T. Bonnesen and W. Fenchel, *Theory of convex bodies*, translated from the German into English, and in 1988 Burago and Zalgaller's monograph *Geometric inequalities* was translated into English from the original Russian. Neither one of these translations may be considered a substitute for the excellent detailed monograph written by Rolf Schneider. I recommend this book to everyone who appreciates the beauty of convexity theory or who uses the strength of geometric inequalities, and to any expert who needs a reliable reference book for his/her research.

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