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Hilbert space operators in quantum physics, by Jiří Blank, Pavel Exner, and Miloslav Havlíček. American Institute of Physics Translation Series in Computational & Applied Mathematical Physics, American Institute Physics, New York, 1994, xiii+593 pp., $75.00. ISBN 1-56396-142-3

Quantum mechanics is not so much a physical theory as a framework for physical theories. In one context it describes electrons and nuclei obeying nonrelativistic dynamics and interacting via electrostatic forces. This is already enough to give a reasonable description of the properties of ordinary matter. Somewhat closer to the frontier of physical theory are relativistic quantum fields that describe quarks (which make up the particles that form the nuclei) and leptons (including electrons). These interact through the medium of other relativistic quantum fields, the gauge fields. An ultimate theory of nature may extend quantum mechanics to incorporate gravity or some yet unknown physical effect. However, in every case, whatever the particles and whatever the forces, quantum mechanics is supposed to provide the framework.

Must this be so? The physicist Steven Weinberg poses this question in his recent popular book [1]. He considers the validity of quantum mechanics as subject to experimental test, but he speculates that "quantum mechanics may survive not merely as an approximation to a deeper truth ... but as a precisely valid feature of the final theory."

Quantum mechanics is formulated in abstract mathematics, and there seems to be no simple pictorial explanation of quantum concepts. If quantum mechanics is ultimate truth, then this is good news for mathematicians—we can deal with abstraction—but bad news for almost everyone else.

The mathematics is that of Hilbert space. A *Hilbert space* is a complex vector space with an inner product that makes it into a complete metric space. Every closed subspace M of a Hilbert space $\mathscr{H}$ is also a Hilbert space, and its orthogonal complement $M^\perp$ is also a closed subspace. The projection theorem says that for every vector ψ in the Hilbert space and for every closed subspace M, there is an orthogonal projection of ψ onto M. This is the unique vector ψ' such that $\psi = \psi' + \psi''$ with ψ' in M and ψ'' in $M^\perp$. The *projection operator* E_M onto M is defined by $E_M\psi = \psi'$. The projection operator $E_{M^\perp}$ onto the orthogonal complement $M^\perp$ is then given by $E_{M^\perp}\psi = \psi''$.

In quantum mechanics the *state* of a system is determined by a unit vector ψ in a Hilbert space $\mathscr{H}$. Each *quantum event* corresponds to a closed subspace M of the Hilbert space $\mathscr{H}$. The fundamental postulate of quantum mechanics is that the probability of the event M when the state is given by ψ is the square $\|E_M\psi\|^2$ of the length of the projection of ψ onto M.

The orthogonal complement $M^\perp$ of M is another quantum event, the complementary event. Then

$$\psi = E_M\psi + E_{M^\perp}\psi \tag{1}$$

represents the vector ψ as the sum of two orthogonal vectors. By the theorem of Pythagoras

$$\|E_M\psi\|^2 + \|E_{M^\perp}\psi\|^2 = \|\psi\|^2 = 1. \tag{2}$$

The probability of an event and of its complement add up to one.

Quantum events M and N are *compatible* if the Hilbert space is the direct sum of the orthogonal spaces $M\cap N$, $M\cap N^\perp$, $M^\perp\cap N$, and $M^\perp\cap N^\perp$. In this case it is meaningful to interpret the quantum event $M\cap N$ as the *conjunction* of the quantum events M and N and to give a similar interpretation for the other pairs. The compatibility condition ensures that the probabilities add up to one.

In general, a collection of quantum events is compatible if every pair is compatible. Each collection of compatible quantum events has the same structure as a collection of events in ordinary probability. These events are associated with some possible measurement. When this measurement is performed, each event may or may not happen. The frequency with which it happens over the long run is predicted by the probability calculation.

The difference between quantum mechanics and ordinary probability is that quantum events are generally not compatible. The usual explanation for this is that the act of performing a measurement to see whether M occurs may preclude the act of performing a measurement on the same system to see whether N occurs, and vice versa. The most familiar example of two incompatible quantum events is where one corresponds to the position of a particle being in a certain bounded set and the other to its momentum being in some other bounded set. Of course the fact that quantum events can be incompatible does not mean that they cannot be measured individually. One can take many separate copies of the system, all prepared in state ψ, and on some of these measure M and on the others measure N. That way one can confirm the probability predictions of quantum mechanics, even for events that are not compatible.

An isomorphism U of a complex Hilbert space $\mathscr{H}$ to itself is called a *unitary operator*. If we apply the isomorphism to both the state vector ψ and the subspaces M, we obtain the same probabilities. This is the *unitary invariance* property of quantum mechanics. If we want to describe probability predictions that change, then we may apply a suitable unitary operator U to the state ψ and leave the subspace M unchanged. This is the *Schrödinger picture*. Alternatively, one may apply the inverse operator U^{-1} to the subspace M, leaving the state ψ unchanged, and obtain the same results. This *Heisenberg picture* is the point of view that is most common in quantum field theory.

The relevant subspaces M and unitary operators U are ultimately defined in terms of the geometry of Hilbert space. The simplest example is that of particle position. Define *position space* to be three-dimensional Euclidean space. A region (Borel subset) R of position space is called a *position event*. For each position event R there is a corresponding quantum event $M(R)$. The correspondence between R and $M(R)$ sends disjoint regions to orthogonal subspaces, and it sends the union of disjoint regions to the direct sum of the subspaces. It sends the empty set to the zero subspace and the whole Euclidean space to $\mathscr{H}$. In addition, for each space translation vector a there is a corresponding unitary operator $V(a)$. The relation between these objects is

$$V(a)M(R) = M(R+a). \tag{3}$$

This relation is a powerful constraint on both the unitary operators $V(a)$ and the subspaces $M(R)$.

The central subject in the text under review is dynamics, that is, the determination of the time evolution. One proceeds from hypotheses about forces to a specification and ultimately a computation of the appropriate unitary time evolution operator $U(t)$ corresponding to an advance in time by t.

Dynamics provides a classification of states. Consider the family $M(R)$ for bounded regions R. Each quantum event $M(R)$ is a closed subspace of the Hilbert space $\mathscr{H}$. A state ψ is a *scattering state* if for every bounded region R

$$\lim_{t\to\pm\infty} \|E_{M(R)}U(t)\psi\|^2 = 0. \tag{4}$$

This says that the particle eventually leaves every bounded region. A state ϕ is a *bound state* if for every $\varepsilon > 0$ there exists a bounded region R such that for all t

$$\|E_{M(R)}U(t)\phi\|^2 \geq 1 - \varepsilon. \tag{5}$$

This says that the particle is essentially confined to a bounded region. The collection of all scattering states and the collection of all bound states each generate closed subspaces, and these two spaces are orthogonal. The book proceeds with the fascinating task of finding conditions on the dynamical evolution that guarantee that these spaces span $\mathscr{H}$. This would say that every state can be decomposed in terms of a scattering state and a bound state. This classification is a beginning toward an understanding of quantum dynamics.

Must everything be so abstract? Could one not represent the Hilbert space as a space of square-integrable complex functions? This is exactly what is done in expositions of elementary chemistry. The Hilbert space is a space of "wave functions". The subspace of functions with support in some measurable subset R is a closed subspace, the event that the position of the electron is in this subset. However, the concrete representation violates the spirit of unitary invariance. There are other quantum events that do not have a simple form in this representation as functions. Ultimately, according to quantum orthodoxy, no preference should be given to one representation over another.

One example of a quantum event that does not have a simple form in the wave function representation is the event that the momentum of the electron is in a set Λ. This event is the subspace of square-integrable functions that can be written in a Fourier expansion using frequencies in the set $\frac{1}{h}\Lambda$. In this formula h is Planck's constant, the characteristic indication of quantum phenomena. It is not supposed to be meaningful to ask simultaneously whether the position is in R and the momentum is in Λ. These events are not compatible. This contrasts with the situation in classical mechanics, where the event that the position is in R and the momentum is in Λ is perfectly well defined.

Classical mechanics is the limit of quantum mechanics when Planck's constant approaches zero. (Of course, this is a mathematical statement—after all, Planck's constant is a constant.) This *classical limit* is subtle because Planck's constant often appears in a denominator, as in the example above.

Physicists and mathematicians often speak of a process of *quantization*, in which one goes in the opposite direction, from classical mechanics to quantum mechanics. In general this is not a unique process, except perhaps in the presence of certain symmetries. Some of these difficulties are reviewed in the book. There is, of course, no reason to believe that quantization should be a meaningful process in general. The correct theory of nature is quantum theory or some variant, and there is no reason that this theory should arise by some systematic process from a degenerate limiting case.

The book by Blank, Exner, and Havlíček reviewed here begins with the necessary background on operators acting in Hilbert space. The remaining chapters are a rather thorough exposition of the mathematical framework of quantum mechanics and of (mainly non-relativistic) quantum dynamics. They cover various foundational questions but touch only very briefly on quantum field theory. The culminating chapters are on Schrödinger operators and scattering theory. This subject is already treated in various other treatises [2–12], but the book under review gives a useful overview. It could be a good text for a first course in the mathematics of quantum mechanics. There are twenty-four pages of references at the end; these should provide an entry into the literature of the subject.

The theory of relativistic quantum fields is supposed to proceed within the framework of quantum mechanics. In this approach the fundamental interactions are between the fields, and the particle description emerges only in scattering limit. The number of particles coming out of a scattering experiment may differ from the number that went in. This is a phenomenon that is not described at all by the non-relativistic theory. The book under review describes the Gårding-Wightman axioms, which are a set of requirements that a relativistic quantum field theory should obey. The attempt to produce non-trivial examples of quantum fields satisfying these axioms is the program of *constructive field theory*. This program has proved remarkably difficult.

Part of the problem is that earlier attempts at constructive field theory dealt with Bose fields [13, 14]. Such fields are the quantum analog of ordinary scalar and vector fields, and so they are rather natural mathematical objects. Nevertheless, it is not clear that they exist in four-dimensional space-time [15]. This does not particularly worry physicists who believe that the fundamental particles are quarks or leptons described by Fermi fields or photons and other particles described by gauge fields. However, these other types of fields are rather strange objects. For instance, gauge fields are the quantum analogs of connections. Scalars and vectors are defined at individual points, but the role of a connection is to define parallel transport along a path between different points. The technical difficulties with a quantum theory of connections are formidable [16].

It is tempting to speculate on issues that are not mentioned in the book. What is the future of quantum mechanics? Will it remain forever the framework for physics? Even if it is not the final framework, it will be difficult to break free. Quantum mechanics works spectacularly well, and so much of current thinking in physics is so deeply imbedded in this framework that extricating oneself will be very difficult. The fundamental postulate of the theory is the theorem of Pythagoras, and what could be more mathematically seductive?

The liberation from quantum mechanics, if it ever comes, may begin with a liberation from unitary invariance. Some quantum events in the theory will become non-events from a physical point of view. Of course a multitude of detailed predictions of quantum mechanics will still be valid, but they will be interpreted as limiting cases of the new theory.

If unitary invariance is discarded, then what representation should be chosen? Some physicists speculate that the representation as functions of position is fundamental. After all, there is a view that all measurements can be reduced to position measurements. This is supported by the fact that momentum measurements in scattering experiments reduce to position measurements. Some attempts have been made along these lines; for example, the de Broglie-Bohm deterministic theory [17, 18] and the Fényes-Nelson stochastic theory [19, 20] both give a fundamental role to the particle representation.

There is even an argument internal to quantum mechanics that the position representation (or some field theory analog) may be preferred. The Feynman path integral has a very appealing structure in the position representation. Say that ψ is the wave function at time zero and that $U(t)\psi$ is the wave function at time t. Let ϕ be another wave function. Then the inner product of ϕ with $U(t)\psi$ is given by

$$\langle \phi, U(t)\psi \rangle = \int \psi(\gamma(0)) \exp\left(\frac{2\pi i S(\gamma)}{h}\right) \overline{\phi(\gamma(t))} D\gamma. \tag{6}$$

The integral is over all functions γ from $[0, t]$ to position space. The quantity $S(\gamma)$ is the classical action of the path γ. The $D\gamma$ represents a multiple of Lebesgue measure on the space of paths. Of course this Lebesgue measure does not exist, so the interpretation of this expression is delicate. But it is beautiful, and it suggests that one representation may be more natural than the others.

Why would one even want to consider moving beyond quantum mechanics? In spite of the spectacular success of quantum mechanics, there is a residual discomfort about the subject [21]. One reason for this is that there is a rule for another kind of state change, other than by the unitary evolution operators $U(t)$. This is stated or explained away in various ways by various authors. Usually this occurs only in passing, as if there were a certain embarrassment about the subject. In the book under review, it is stated that when the measurement is made to see whether the event given by M has occurred and the result is yes (which happens with probability $\|E_M\psi\|^2$), then the state vector changes from ψ to $E_M\psi/\|E_M\psi\|$. Most physicists are convinced that this sort of prescription is ad hoc and too simple, but they do not seem to know what to do about it, though various proposals (consistent histories, decoherence) have been made [22]. This is part of a general feeling that the physical interpretation of quantum mechanics is mysterious. In the words of Weinberg: "I admit to some discomfort in working all my life in a theoretical framework that no one fully understands."

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Clifford algebra and spinor-valued functions, a function theory for the Dirac operator, by R. Delanghe, F. Sommen, and V. Souček. Kluwer, Dordrecht, 1992, xvi + 485 pp., $176.00. ISBN 0-7923-0229-X

The seminal paper in which W. K. Clifford introduced the algebras which now bear his name appeared in the first volume of the *American Journal of*