Acknowledgment

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About the title

Among all combinations between the basic nouns of mathematics (algebra, analysis, geometry, topology, ...) and the corresponding adjectives, the grouping *geometric analysis* seems to be one of the most recently coined. Trying to define it might be difficult and unnecessarily reducing. But in the book under review (as for its predecessor *Groups and geometric analysis* by the same author [6]), "analysis" means study of differential operators and integral transforms, and "geometric" is to be understood as related to a group action. Thus the book discusses in depth such topics as Radon transforms, (generalized) Fourier transforms, invariant differential operators on homogeneous spaces of Lie groups, and links of these objects with representation theory. It cannot be considered as a textbook on representations, however, no attempt being made here at a systematic treatment of the subject.

Symmetric spaces (and some related spaces) make up the framework of the book almost exclusively, and this choice calls for some comments. Symmetric spaces were born seventy years ago as a nice, but rather anecdotal, exercise in Riemannian geometry: find all Riemannian manifolds the curvature tensor of which is invariant under parallel transport. In a short 1926 note, Harry Levy gave a partial solution, including the case of spaces with constant curvature. This note urged Elie Cartan then to publish his own independent research on the same question, and Cartan quickly became, and remained for years, the master of this "classe remarquable d'espaces de Riemann" [2]. Renamed "symmetric spaces" in 1929, i.e. Riemannian manifolds where all geodesic symmetries are isometries, they revealed deep links with many fields of mathematics: semisimple Lie groups (leading to a classification), topology, analysis, automorphic forms , ...

For the purpose of the present book, a symmetric space can be conveniently defined as a homogeneous space $X = G/K$ where G is a real Lie group and K is the fixed point subgroup of an involutive automorphism of G. The group K is assumed to be compact throughout, whence a G-invariant Riemannian metric on X. Basic examples are the hyperbolic spaces $H^n = SO_0(1, n)/SO(n)$, the spheres $S^n = SO(n+1)/SO(n)$ and the Euclidean spaces $\mathbb{R}^n = M(n)/O(n)$ where $M(n)$ is the Euclidean motion group, i.e. the spaces with negative, positive or zero constant curvature. The major part of Helgason's book deals with *Riemannian symmetric spaces of the noncompact type*, i.e. the case (generalizing H^n) when G is a connected noncompact semisimple Lie group with finite centre and K is a maximal compact subgroup. It should be noted that non-Riemannian symmetric spaces, the field of active current research in representation theory, are not considered here.

Semisimple Lie groups have a rich structure theory developed by Cartan, Harish-Chandra, Helgason and many others, arising from the root system in the Lie algebra. This fosters the search for fully explicit answers to all problems under study, which can indeed be achieved by an elegant (though hard, at times) blend of algebra, analysis and differential geometry. Such methods and results are the subject of the book. Readers are clearly expected to have some familiarity with Lie groups, differential operators and with the semisimple machinery (roots, Iwasawa decomposition, invariant measures, spherical functions) as can be gained for instance from Helgason's previous volumes, [5] and [6].

We now survey some of the contents in more detail.

Radon transforms

Nearly half of the book is devoted to Radon transforms (Chapters I, II and IV), reflecting the author's interest and work over many years. To start with the simplest example, let f be a function on the plane, and let

$$Rf(l) = \int_{x \in l} f(x)\, dm_l(x)$$

denote its integral over all points x belonging to the line l, with measure m_l induced by the Euclidean measure of $\mathbb{R}^2$. The first problem is to recover f from its *Radon transform* Rf. A natural guess (though not correct!) is that the value $f(x)$ can be obtained from the lines l passing through x, and one introduces the *dual Radon transform* of a function φ on the set of lines

$$R^*\varphi(x) = \int_{l \ni x} \varphi(l)\, d\mu_x(l),$$

integrating φ over all lines l through the point x with the natural angular measure μ_x. Thus R^* is dual to R in the sense of projective duality between lines and points; furthermore, R and R^* are dual operators too:

$$\int_{\mathbb{R}^2} f(x) R^*\varphi(x)\, dx = \int_{\mathscr{L}} Rf(l)\varphi(l)\, dl,$$

where $\mathscr{L}$ is the set of lines with its natural measure dl.

Now the same isometry group $M(2)$ acts transitively on both $\mathbb{R}^2$ and $\mathscr{L}$. This observation led Helgason to consider much more general *homogeneous spaces in duality* $X = G/K$ (replacing $\mathbb{R}^2$) and $\Xi = G/H$ (replacing $\mathscr{L}$), where G is a Lie group (replacing $M(2)$) and H, K are two closed subgroups. The basic relation $x \in l$ is then replaced by the more general *incidence* relation: the elements x of X and ξ are Ξ are incident if as cosets in G they are not disjoint. This yields a double fibration:

$$\begin{array}{ccc} & Z = G/K \cap H & \\ \swarrow & & \searrow \\ X = G/K & & \Xi = G/H, \end{array}$$

where Z identifies to the set of all incident couples (x, ξ). Under some mild assumptions on the groups, one can define dual integral transforms R resp. R^* (denoted $^\wedge$ and $^\vee$ in the book), integrating over all x incident to ξ resp. all ξ incident to x. In group terms

$$Rf(gH) = \int_{H/K\cap H} f(ghK)\, d\dot{h}, \qquad R^*\varphi(gK) = \int_{K/K\cap H} \varphi(gkH)\, d\dot{k},$$

where $d\dot{h}$ and $d\dot{k}$ are natural invariant measures.

Three major problems are studied:

A) *Inversion problem*: Recover the function f from its Radon transform Rf.
B) *Range problem*: Find the ranges and kernels of R and R^* acting on certain function spaces.
C) *Support problem*: Find information on $\operatorname{supp} f$ from information on $\operatorname{supp} Rf$.

At this level of generality the scope of these questions seems too wide, however, for one to hope to unify all cases in the answers. One must restrict to more specific situations, suggested by geometric or group theoretic considerations. What are, for instance, natural substitutes for the Euclidean lines in the hyperbolic unit disk $X = H^2 = SU(1, 1)/SO(2)$? First answer: take as Ξ the set of all *geodesics* of X (circles orthogonal to the unit circle). The corresponding Radon transform may be called a generalized X-ray transform, recalling its motivation from the mathematical theory of tomography (where problems A and C are of great practical importance). A second answer is: take as Ξ the set of all *horocycles* of X, that is, the "wave surfaces" orthogonal to a "parallel beam of rays" (geodesics meeting at infinity on the unit circle). The horocycles are thus all circles inwardly tangent to the unit circle.

Both settings are considered in the book; more generally, Helgason deals with the following Radon transforms:

- integration over k-dimensional totally geodesic submanifolds in a space with constant curvature: $\mathbb{R}^n$, H^n, S^n ("geodesic transform");
- Radon transform for Grassmannians, with a specific incidence relation between p-planes and q-planes in $\mathbb{R}^{p+q+1}$;
- Poisson integrals for bounded domains: for example, if $G = SU(1, 1)$ acting on the complex plane and if K and H are the isotropy subgroups of the points 1 and 0 respectively, then X is the unit circle, Ξ is the unit disk, and Rf is the classical Poisson integral of f;
- integration over horocycles in a Riemannian symmetric space of the noncompact type ("horocycle transform");
- an analogue of the latter on the tangent space X_0 to X at the origin ("flat horocycle transform").

The horocycle cases make up one third of the book, because of their links with the semisimple arsenal, Fourier analysis and representation theory. The tangent space appears as an interesting limit case in the following way. Think of an inflating sphere S^2, with fixed North Pole 0 and its centre going to infinity; on the tangent space $\mathbb{R}^2$ at 0 the isometry group (translations/rotations) appears as the limit (contraction) of the isometry group of S^2 (rotations around axes meeting/not meeting the equator). Similarly, Euclidean geometry and harmonic analysis in $\mathbb{R}^2$ appear as limit cases of their analogues for a hyperbolic disk with increasing radius. But surprisingly enough, tangent space analysis is not easier than its curved analogue; some results for X_0 are indeed obtained as limits from X...

Typical Radon inversion formulas (problem A) are

$$f = R^*ARf, \quad \text{or} \quad f = BR^*Rf,$$

where A, B are certain explicit integro-differential operators (involving Hilbert transforms, or Riesz operators, etc.), reducing to plain differential operators under some evenness assumptions. They do not imply support theorems easily, however, and solving problem C often requires such harder tools as a Paley-Wiener theorem. Problem B is of particular interest when $\dim X < \dim \Xi$ and the range of R can be characterized as the kernel of a certain differential operator on Ξ. This extends the Poisson integral example quoted above, where X is the circle, Ξ is the disk, and the range of R is known to be the kernel of the Laplace operator on the disk.

A different approach to Radon transforms originated in the work of Gel′fand, Guillemin and Sternberg [4], who started from a double fibration between general manifolds X, Ξ and Z. This led in particular to remarkably precise support theorems, obtained by means of Fourier integral operators and analytic wavefront sets (see e.g. Boman and Quinto [1]). No group theory is involved here, and this approach lies outside the spirit of Helgason's book, where it is but briefly mentioned.

Fourier transform

Chapter III, the thickest of the book, develops Fourier analysis on a Riemannian symmetric space of the noncompact type $X = G/K$. The Fourier transform of a smooth compactly supported function f on X is

$$\tilde{f}(\lambda, b) = \int_X f(x) e^{\langle -i\lambda+\rho, A(x,b)\rangle} dx.$$

It is inverted by

$$f(x) = \int_{\mathfrak{a}^* \times B} \tilde{f}(\lambda, b) e^{\langle i\lambda+\rho, A(x,b)\rangle} |c(\lambda)|^{-2} d\lambda\, db.$$

Some significant features of the formulas can be grasped without explaining the notation in full detail. First, this is a scalar-valued transform, unlike the operator-valued Fourier transform for G given by representation theory. The dual variable (λ, b), with λ in the finite dimensional vector space $\mathfrak{a}^*$ and b in the boundary B of X (B = unit circle for $X = H^2$), is similar to the polar coordinates couple (r, θ). Thus $\tilde{f}$ is a non-Euclidean analogue of the usual Fourier transform written in polar coordinates. As a function of x, the exponential is an *eigenfunction of all G-invariant differential operators* on X, showing that this is Fourier analysis indeed. In the exponent, $A(x, b)$ is the (vector-valued) "distance" from the origin to the horocycle passing through x and the point b at infinity. Horocycles are thus level surfaces of $A(x, b)$ for fixed b, and it follows easily that $\tilde{f}$ *is a Euclidean Fourier transform of the horocycle Radon transform Rf.* This observation acts as cement between different chapters of the book. Finally, the Plancherel measure $|c(\lambda)|^{-2} d\lambda\, db$ involves Harish-Chandra's celebrated, and explicitly known, c-function.

Another interesting remark is that the Poisson kernel for the unit disk is an exponential of the function $A(x, b)$; thus the classical Poisson integral can be viewed as a formula in non-Euclidean Fourier analysis.

In this substantial chapter, one will find an asymptotic study of Eisenstein integrals, complete proofs of Paley-Wiener theorems for the Fourier transform, and a theory of its tangent space analogue. *Spherical functions* are of course closely related to the above theory, as K-invariant eigenfunctions on X. Though not a symmetric space itself, the space Ξ of horocycles in X bears some striking analogies with X. This led Helgason to develop a parallel theory of *conical functions and distributions*, the Ξ-analogue of spherical functions, with interesting applications to representations of the group G.

For the K-invariant L^p-type theory on X, not included here for brevity, the reader is referred to Gangolli and Varadarajan [3].

Concluding remarks

The above survey does little justice to the rich contents of the book, also discussing global solvability of invariant differential operators, harmonic functions, Poisson transforms, Shilov boundaries, Hua operators, wave propagation and Huygens' principle on symmetric spaces, irreducibility of eigenspace representations...

Most material is taken from Helgason's papers over thirty years, not previously available in book form, with many recent additions or simplifications from other authors. This work is the promised sequel to [6], and I would suggest reading it at two levels: 1) main definitions and theorems, enlightened by their numerous inspiring comments and remarks; 2) details of the proofs, with the predecessor volume [6] at hand. Many further results are given as exercises, with solutions or references.

Besides, I found no serious lacuna in the index and very few misprints in the book, none of them bothersome. (At the author's request, I mention that the remark on page 274 should be deleted.) As in [5 and 6], for which Helgason received the 1988 Steele Prize for expository writing, the style is very fluent and pleasant, conducting the reader at a regular pace. I think the present book will be a most valuable (and reasonably priced) reference for anyone interested in Radon transforms and analysis on semisimple Lie groups.

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Traveling wave solutions of parabolic systems, by Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert. Transl. Math. Monographs, vol. 140, Amer. Math. Soc., Providence, RI, 1994, xii + 448 pp., $142.00. ISBN 0-8218-4609-4

The theory of systems of nonlinear parabolic P.D.E. is a centerpiece of modern applied mathematics, and such equations have a virtually ubiquitous presence as mathematical models in science and engineering. Such systems often assume the form

$$u_t = D\Delta u + f(u, \nabla u),$$

where $u \in \mathbb{R}^n$ and D is an appropriate matrix. When the spatial variable x is one-dimensional and unrestricted, the equations frequently admit travelling waves, which are special translation invariant solutions of the form $u(x, t) = U(x - ct)$, where $U = U(\xi)$ is the profile of the wave which propagates through the one-dimensional spatial domain at constant velocity c. Such solutions are studied as a paradigm for the behavior exhibited in many model problems, and