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Quantum invariants of knots and 3*-manifolds*, by V. G. Turaev, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter, Berlin, 1994, x + 588 pp., $118.95 (DM 288), ISBN 3-11-013704-6

"In the last decade we have witnessed the birth of a fascinating new mathematical theory. It is often called by algebraists the theory of quantum groups and by topologists quantum topology. These terms, however, seem to be too restrictive and do not convey the breadth of this new domain..." So begins this book, and so began this field about ten years ago with the discoveries of Jones, Drinfel′d, Witten, and several other pioneers, among them the author of this book. The field was born with the discovery that for every simple Lie algebra $\mathfrak{g}$, there is a Hopf algebra $U_q(\mathfrak{g})$, or quantum group, and a link invariant which is a polynomial in q. If q is a root of unity, the link invariant extends to a 3-manifold invariant. The Jones and HOMFLY polynomials are prime examples.

The excitement peaked in the late 1980's. Algebraists and physicists had found completely new topological invariants, invariants that were both mathematically deep and had simple definitions (sometimes nearly trivial definitions, as in the case of Kauffman's beautiful definition of the Jones polynomial via the Kauffman bracket [4]). It was then up to enterprising topologists to use them to spin out results about the structure of knots and 3-manifolds. The Kauffman-Murasugi theorem establishing the crossing number of alternating knots via the Jones polynomial was surely a good first step.

However, quantum 3-manifold topology has not yet lived up to its expectations. Although it is important to algebraists and physicists, it has so far produced very little new 3-manifold topology. By contrast, Donaldson theory, which may now also be called quantum topology due to the work of Seiberg and Witten, is fundamental to 4-manifold topology. The Casson invariant, which is closely related to Donaldson theory, has also been useful in 3-manifold topology. Yet even the definitions in quantum 3-manifold topology are cluttered with loose ends, formalism, and complications. Often, simple-looking definitions become much more complicated when they are made vigorous.

For example, the author's approach, which originates with his work with Reshetikhin, is to define invariants $\tau(M)$ of 3-manifolds M using framed links and modular categories, where the relevant example of a modular category is a kind of semisimple quotient, or Jacobson radical, of the representation theory of a quantum group at a root of unity. Understanding that representation theory involves cataloguing the irreducible representations and how they tensor. Dividing by the radical is no mean feat, either: It is a deep result of Andersen [1] (and, independently, Turaev and Wenzl for classical Lie algebras [9]) that the quotient exists for every $\mathfrak{g}$. Andersen's approach is to analyze certain indecomposable modules in the non-semisimple representation theory as well as the irreducible modules. Turaev and Wenzl's approach is to rely on the full combinatorics of classical Lie representations, including Young tableaux and the Littlewood-Richardson rule. Do we really need so much algebra to interpret Witten's famous formula [10],

$$\tau(M) = \int_{\mathcal{A}} e^{\frac{ik}{4\pi}\int_M Tr(dA\wedge A+\frac{2}{3}A\wedge A\wedge A)} d\mathcal{A}, \tag{1}$$

for the same or very similar invariants? (In this formula, which is known as the Chern-Simons path integral, $\mathcal{A}$ is the moduli space of connections on a G-bundle over M, where G is the connected, simply-connected, compact Lie group with (real) Lie algebra $\mathfrak{g}$, and A is a $\mathfrak{g}$-valued 1-form associated to such a connection.) On the other hand, equation (1) is only meaningful because it is "renormalizable", "gauge-invariant", and "anomaly-free". Moreover, "regularizing" the path integral involves choosing a framing of the 3-manifold, which is a hidden extra structure. Perhaps equation (1) is not so simple after all; understanding its full meaning is one of the main problems in quantum 3-manifold topology.

Nevertheless, there is no substitute for straightening out the details. For this reason, this book is a fundamental contribution to quantum topology. It is excellent as a compilation of existing results; it covers many definitions and theorems that are independently rediscovered all too often, such as the Turaev-Viro state model on triangulations of 3-manifolds [8]. Moreover, the author has added his new and important theory of shadows to this survey. It is also a reliable text for those who want to learn many of the standard topological arguments and constructions. In general the topology is treated in detail, while the algebra is usually summarized or referenced.

The first part of the book gives a category-theoretic approach to two main invariants in 3-dimensional quantum topology, the Reshetikhin-Turaev ribbon graph invariants and the Jones-Reshetikhin-Turaev-Witten invariants of 3-manifolds, and it discusses topological quantum field theories (TQFT's), which are the comrades-in-arms of the 3-manifold invariants. It begins with a quick definition of both invariants using Reidemeister moves in the former case and Dehn surgery and Kirby moves in the latter case, following two papers by the author and Reshetikhin [6, 7]. Although this is underemphasized in the book, the graph invariants include the usual Jones, HOMFLY, and Kauffman polynomials of links, which correspond to the cases where $\mathfrak{g}$ is, respectively, sl(2), sl(n), and so(n) and sp($2n$) minus those representations of the former that do not extend to $O(n)$. Recall that, by Atiyah's definition, a TQFT is a functor from the category of surfaces with bordisms as morphisms and geometric reversal as duality to the category of finite-dimensional vector spaces with usual duals and adjoints. The book proceeds to give this definition with care, and with even greater care, explains that many of the invariants involve a central extension of the category of bordisms given by framings of bordisms and 3-manifolds minus the spin structures.

The second part of the book is the most interesting to me personally. It defines the Racah-Wigner $6j$-symbols that originated with particle-spin computations in mathematical physics in the context of "unimodal categories". These categories are like ribbon categories, but without the structure that defines crossings, so that they only yield invariants of planar graphs. The $6j$-symbols satisfy the Biedenharn-Elliot relation, which comes from associativity of the tensor product, and this relation in turn leads to the topological invariance of a state sum model on 3-manifolds. (Indeed, some old treatments of $6j$ symbols from mathematical physics use diagrams of tetrahedra for both the symbols and the relation that correspond exactly to combinatorial moves on triangulations.) This is the generalized Turaev-Viro invariant $|M|$. The sl(2) case was the first partial answer to Atiyah's question of finding a state model invariant on a triangulation of a 3-manifold that might in some sense converge to the Chern-Simons path integral as the triangulation is refined. (A state model is a discrete version of a quantum field theory and is widely used in statistical mechanics.) However, $|M|$ and $\tau(M)$ differ; rather,

$$|M| = \tau(M)\tau(-M). \tag{2}$$

This section proceeds to the author's exciting further progress on Atiyah's question: $\tau(M)$ can be defined by a state model on the 2-skeleton of a 4-manifold W with $\partial W = M$. These 2-skeletons are a special case of what are called "shadows" in the book. In particular, the Turaev-Viro model arises when $W = M \times I$, which yields a natural geometric explanation of equation (2).

The third part of the book reviews the construction of ribbon categories from the quantum groups $U_q(\mathfrak{g})$ and modular categories from the same quantum groups when q is a root of unity. It is an adequate survey of the algebraic results that a topologist needs for explicit computations. The sophisticated results of Andersen mentioned above, as well as the equally sophisticated results of Lusztig that Andersen uses, are only referenced. Many other algebraic results on quantum groups are not mentioned. But it is natural that this section is less complete, since a more thorough treatment is the subject of another entire book [5]. The book gives a good graphical treatment of the representation theory of $U_q(\mathrm{sl}(2))$ which originates with the Temperely-Lieb algebra and which has been advanced by Kauffman, Lickorish, Mausbaum, Vogel, and others.

The book's definition of invariants from shadows could help solve some of the main problems of quantum 3-manifold topology. The Reshetikhin-Turaev definition of $\tau(M)$ does not really answer Atiyah's question even though it is in some sense a state model, because Dehn surgery is not a local construction. Mysteriously, direct attempts at constructing local state models for $\tau(M)$ using modular categories have failed. The explanation may lie in the Chern-Simons path integral. The Chern-Simons 3-form

$$\omega_A = A \wedge dA + \frac{2}{3} A \wedge A \wedge A,$$

unlike the Yang-Mills form, cannot be locally gauge invariant, because there exist global gauge transformations that change its value by an integer. It therefore cannot be discretized with purely invariant data such as tensors in a modular category. However, if $F_A = dA + A \wedge A$ is the curvature of A, then on a 4-manifold W,

$$d\omega_A = Tr(F_A \wedge F_A),$$

which is locally gauge-invariant. If $M = \partial W$, then by Stokes' theorem,

$$\int_W Tr(F_A \wedge F_A) = \int_M \omega_A.$$

Perhaps there is a corresponding path integral which is also the continuous limit of the Turaev state model. It would be a quantum field theory which only gives topological information about the boundary. This may also yield an alternate explanation of why framings arise in Witten's definition, since every fourth cobordism class (or signature) of W can be identified with every third framing of M if M is given any fixed spin structure.

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