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Affine differential geometry, by Katsumi Nomizu & Takeshi Sasaki, Cambridge Tracts in Math., vol. 111, Cambridge University Press, Cambridge, 1994, xiv + 263 pp., $49.95, ISBN 0-521-44177-3

Affine differential geometry had its heyday in the twenties; witness the monograph *Vorlesungen über Differential Geometrie* II, *Affine Differentialgeometrie* by W. Blaschke in collaboration with K. Reidemeister. To give the uninitiated reader an idea of the subject, consider an immersion $f : M \to A^3$ of an orientable surface M into the affine 3-space A^3. Fixing a volume element on A^3, the group of structure preserving automorphism is the equi-affine group $Aff_1(A^3)$, formed by the translations combined with the operation of an element of the special linear group $SL(3)$. Let ζ be a vector field along f such that $\zeta(p)$ is transversal to the tangent plane T_pf of f at p. The flat covariant derivation D on A^3 then yields on M a torsion free covariant derivation ∇ on M by the formula

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X, Y)$$

where $f_* : T_pM \to T_pf \subset T_{f(p)}A^3$ is the canonical inclusion.

$\nabla_X Y$ is the projection of $D_X f_*(Y)$ into T_pf along the "normal vector" $\zeta(p)$. $h(X, Y)$ is a symmetric bilinear form. It is essentially the second fundamental form of surface theory in euclidean 3-space.

Of course, ∇ will depend on the choice of the normal field ζ. To give ∇ an intrinsic meaning, ζ should be made $Aff_1(A^3)$-equivariant. That means: If $\varphi \in Aff_1(A^3)$, then $\varphi_*\zeta(p) = \zeta(\varphi p)$. The first basic result of affine differential geometry for surfaces in 3-space is that such a field exists and that it is essentially unique, provided the quadratic form h is nondegenerate.

This can be seen as follows: While the tangent plane T_pf is the linear approximation to f near p, the second-order approximation will be a paraboloid. In case h is definite, it will be an elliptic paraboloid. Now take $\zeta(p)$ in the axis of this paraboloid. In case h is indefinite (and nondegenerate), it will be a hyperbolic paraboloid, again determining by its axis a transversal direction. Clearly, these osculating paraboloids are defined $Aff_1(A^3)$-equivariantly.

Actually, if h is nondegenerate, we may view it as a (possibly indefinite) Riemannian metric on M. The canonical Levi Civita connection determined by this metric is the same as the one yielded by the above defined canonical ζ.

So much for one of the basic constructions in affine surface theory. It is a pity that in the book under review a geometric interpretation of the canonical normal field is lacking. In Blaschke's monograph, such an interpretation is given, of course. Maybe the reason for this lack is that the authors believe that geometric arguments do not fit into what they call the "structural point of view" of affine differential geometry. According to the introduction of the book, this point of view was introduced by Nomizu in 1982 in a lecture at Münster University with the grandiose title "What is Affine Differential Geometry?"

In the opinion of the reviewer, the authors vastly overestimate the importance of their point of view. In reality, it is often not much more than the use of a modern terminology. Even worse, the authors seem to be unaware of the fact that some of their basic notions have been anticipated in the literature. Take, as an example, the concept "affine immersion". This is nothing but the "rigging" of a submanifold in an affine space, as used already by Schouten in his book *Ricci calculus*, second edition 1954.

Even more lamentable is the lack of credit given to L. Berwald when the authors introduce the concept of a "Blaschke structure". It should be called "Blaschke-Berwald structure", if it is necessary at all to have such a concept from a "structural point of view". It was Berwald who generalized Blaschke's construction on surfaces in 3-space to hypersurfaces, so Berwald deserves to be remembered. He was a Jew and therefore lost his job in Prague under German occupation. In 1941, he died under miserable circumstances in the ghetto of the city of Lodz.

The main body of the book under review is devoted to hypersurfaces. The presentation is lucid and very readable. In recent years, there has been a new wave of papers devoted to affine differential geometry. After all, there is no limit to the number of questions one may pose. But it also is obvious that the field is isolated from the main stream of mathematics. There is no hard analysis involved, except for a few papers by Calabi, Terng, and Yau. But their proofs fall outside the scope of this monograph. A look at the type of journals in which the papers quoted in the book have appeared also bespeaks the quality of the results presented. An ambitious young mathematician should look elsewhere to find his challenges. On the other hand, for an old pro, this is the ideal field to present his skill. As a fringe benefit for the mathematical community, a nice and honourable piece of mathematics is kept from falling undeservedly into oblivion.

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