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Compatibility, stability and sheaves, by J.L. Bueso, P. Jara, and A. Verschoren, Pure and Appl. Math., vol. 185, Marcel Dekker, New York, 1995, xiv + 265 pp., $125.00, ISBN 0-8247-9589-X

The focus of this book is to construct an appropriate setting in which to do algebraic geometry over noncommutative noetherian rings. The principal tool of "modern" algebraic geometry is the structure sheaf $\mathcal{O}_R$ associated to the topological space $Spec(R)$ of any commutative ring R. The space $Spec(R)$ consists of all prime ideals of R endowed with the Zariski topology, i.e., open subsets are determined by the ideals I of R as follows:

$$X(I) = \{P \in Spec(R) : I \not\subset P\}.$$

For a principal ideal J with generator f, the sheaf associates to the basic open set $X(J)$ the ring of fractions R_f. Here, R_f is the localization of R at the multiplicative set consisting of the powers of f. This can be extended in a natural fashion so that to any open set $X(I)$ one associates the ring of sections, denoted $\Gamma(X(I), \mathcal{O}_R)$. Furthermore, this extension defines a contravariant functor to the category of sheaves of rings. If R is noetherian, then the explicit structure for the ring of sections for an arbitrary open set $X(I)$ is given by Deligne's formula

$$\Gamma(X(I), \mathcal{O}_R) = \varinjlim Hom(I^n, R).$$

The stalk over a point $P \in Spec(R)$ is then R_P, the localization of R at the set-theoretic complement of P.

Several difficulties arise when one attempts to extend this sheaf construction to noncommutative rings. To simplify matters, assume that all noncommutative rings are both left and right noetherian. While there are some "obvious" methods of constructing a presheaf over $Spec(R)$, in general these do not yield a sheaf. Even when restricting to a certain subtopology of the Zariski topology on $Spec(R)$—which insures that the presheaf is a sheaf—the process will not be functorial. Before proceeding to describe how the authors address these issues, it is convenient to review the basic definitions and terminology of noncommutative localization theory.

The process of "inverting" the elements of a multiplicative subset of a commutative ring is a relatively simple one seen in a first year graduate algebra course. The first example one considers, of course, is the construction of the rationals from the ring of integers, the rationals being the set of all "fractions" a/b, a and b integers, $b \neq 0$ with the usual relation $a/b \equiv c/d$ if $ad = bc$. More generally, if S is a multiplicatively closed subset of R, then one forms the ring of fractions of R with respect to S by considering all fractions with the denominator an element of S and defining $a/b \equiv c/d$ if $ad - bc$ is annihilated by an element of S. Addition and multiplication extend in a natural fashion from R to this new set so as to form a ring. The most important example of a multiplicatively closed set is the complement of a prime ideal P in R. Recall that an ideal of a commutative ring is prime precisely when its complement is a multiplicatively closed set. The ring of quotients at such a set is a local ring, i.e., a ring with a unique maximal ideal. Furthermore, noetherian local rings are well understood. Hence this process of forming a ring of fractions with respect to a multiplicative set is commonly called localization (at a multiplicative set).

The relation defined above on the "fractions" is not, in general, a congruence relation when the ring is noncommutative. Hence this process of localization or inverting elements cannot be applied to noncommutative rings to obtain a sheaf over $Spec(R)$. However, an abstract technique of localization due to Gabriel [G] can be used in its place. This method is motivated by the observation that the rational number 2/3 can be thought of as the map from the ideal $3\mathbf{Z}$ to the integers $\mathbf{Z}$ which sends $3 \mapsto 2$. Then two maps are "equal" if, when restricted to the intersection of their domains, they are equal as functions. Hence, if $\mathbf{Q}$ is the ring of rationals and $\mathcal{J}$ the set of nonzero ideals of the integers $\mathbf{Z}$, then

$$\mathbf{Q} \simeq \varinjlim_{J \in \mathcal{J}} Hom(J, \mathbf{Z}).$$

The set of nonzero ideals of $\mathbf{Z}$ is an example of what is called an idempotent or Gabriel filter of left ideals in a ring R (see also [Go1], [Go2] or [S] for the definitions and basic results). If σ is such a filter of left ideals and if $T = \{x \in R : Jx = 0, \text{ for some } J \in \sigma\}$, then one can show that T is an ideal and

$$Q_\sigma(R) = \varinjlim_{J \in \sigma} Hom(J, R/T)$$

is a ring called the localization or ring of quotients of R at σ.

An important example of an idempotent filter is obtained by fixing some two-sided ideal I of R and letting σ_I be the set of ideals that contains some power of I. If R is commutative and $I = \langle f \rangle$ is principal, then the localization of R at σ_I is canonically isomorphic to the localization of R at the multiplicative set consisting of the powers of f. Denote by $Q_I(R)$ the localization of R at σ_I. In a result of interest in its own right that also motivates some of the later work in the book, the authors prove a generalization of Deligne's formula that states that if R is commutative and I a finitely generated ideal (for the moment we need not assume the R is noetherian), then

$$\Gamma(X(I), \mathcal{O}_R) = Q_I(R).$$

With these tools a presheaf over $Spec(R)$ can readily be constructed by associating to each open set $X(I)$ the ring $Q_I(R)$. However, this fails to be a sheaf on $Spec(R)$ because for arbitrary idempotent filters σ and τ over noncommutative rings the localization functors Q_σ and Q_τ need not commute as they do over commutative rings. To remedy this situation the authors consider a subtopology of the Zariski topology (a topology that yields the full Zariski topology on $Spec(R)$ when R is commutative) by using only open sets of the form $X(I)$, where σ_I is a centralizing biradical. The functors Q_I and Q_J will commute in this case, and so one obtains a sheaf over $Spec(R)$. Without going into a precise definition, σ_I will be a centralizing biradical essentially when I satisfies the left and right Artin-Rees property. In fact, these two properties are equivalent with the additional assumption that R satisfies the strong second layer condition. This condition will be discussed in more detail later. An ideal I in R is said to have the (left) *Artin-Rees* property if for any left R-module M and submodule N, given a positive integer n, there exists a positive integer r such that $I^r M \cap N \subset I^n N$. The well-known Artin-Rees Lemma (cf. [Ma]) states that every ideal in a commutative noetherian ring has the Artin-Rees property.

While the subtopology determined by the ideals with the Artin-Rees property appears to be rather small, there are in fact many noncommutative rings that have sufficiently large numbers of ideals satisfying this property, e.g., enveloping algebras, certain group rings, pi rings. As stated above, the presheaf over $Spec(R)$ with this subtopology is a sheaf; unfortunately it is not functorial. Indeed, an arbitrary ring homomorphism $\phi : R \longrightarrow S$ does not even induce a map from $Spec(S)$ to $Spec(R)$, since the inverse image of a prime ideal is not necessarily prime. However, there are large classes of homomorphisms that behave well with respect to prime ideals. In recent years much work has been done on centralizing and strongly normalizing ring extensions. A ring homomorphism $\phi : R \longrightarrow S$ is called *centralizing* if S, as a module over R, is generated by elements that commute with the image of each element of R. It is called *strongly normalizing* if S, as a module over R, is generated by elements that commute with the image of each ideal of R. If ϕ is a centralizing ring homomorphism, then it will induce a continuous function from $Spec(S)$ to $Spec(R)$, assuming they are endowed with the above subtopology. A strongly normalizing extension, which is a weaker condition, will also induce a continuous map between the prime spectrums if R is assumed to have an additional property. This property is again the strong second layer condition, which involves the structure of the indecomposable injective modules. Roughly speaking, R satisfies the strong second layer condition if the second layer of any indecomposable module E is well behaved. By the second layer of E we mean the module $E/N_P(E)$, where P is the prime ideal of R associated to E and $N_P(E) = \{x \in E : Px = 0\}$. The structure of this module is well understood when R is commutative noetherian (see [M]). The second layer and strong second layer condition arose in the study of classical localization theory, i.e., determining when a multiplicative subset S of a noncommutative ring R is sufficiently nice so that the classical localization of R with respect to S exists. By the classical localization we mean a ring $Q(R)$ and a ring map $\varphi : R \longrightarrow Q(R)$ such that (i) $\varphi(s)$ is invertible for each $s \in S$ and (ii) each element of $Q(R)$ can be written in the form $\varphi(s)^{-1}\varphi(r)$ for some $s \in S$ and $r \in R$. See for example [GW] or [J]. One then has the following result on the functorial properties of the sheaf:

Theorem. *Let $\phi : R \longrightarrow S$ be a ring homomorphism between noetherian prime rings. Then ϕ induces a morphism of ringed spaces*

$$(Spec(S), \mathcal{O}_S) \longrightarrow (Spec(R), \mathcal{O}_R),$$

whenever one of the following is true:

1. *ϕ is a centralizing extension,*
2. *ϕ is a strongly normalizing extension and R satisfies the strong second layer condition.*

By taking global sections, the above morphism of ringed spaces will yield back the ring homomorphism $\phi : R \longrightarrow S$.

The book contains a complete and quite readable introduction to both abstract and classical localization theory. In fact, it is somewhat rare to find a book that deals with both of these topics, since most ring theorists tend to work on one of these areas to the exclusion of the other. The authors use tools from both areas to examine the ideal structure of noncommutative rings and the behavior of this structure under localization and ring extensions. Ring theorists working in either abstract or classical localization theory will be interested in this book to see the interplay between the two subjects and how they combine so as to describe a noncommutative setting for algebraic geometry. The book also sheds light on some of the geometry of commutative, non-noetherian rings.

While the book draws on existing papers, many of which were written by the authors themselves, it also contains improvements on the known results along with shorter proofs. The final sections of the book contain new results on structure sheaves and their functorial properties for certain classes of ring extensions. The book is a very accessible introduction to noncommutative algebraic geometry as well as a valuable resource for any mathematician interested in working on this topic.

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