BOOK REVIEWS

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Vorticity and turbulence, by Alexandre Chorin, Appl. Math. Sci. Ser., vol. 103, Springer-Verlag, New York, 1994, viii + 174 pp., $35.00, ISBN 0-387-94197-5

Understanding fully developed turbulence in fluids is important for many applications ranging from large scale geophysical flows in the atmosphere and the ocean to engineering flows involving, for example, aircraft, turbines, and combustion engines. The type of turbulence theory that is most familiar to pure mathematicians as a scientific discipline involves the transition to turbulence where there are only a few degrees of freedom describing the process of instability, and techniques and concepts apply from the theory of finite dimensional ordinary differential equations such as bifurcation, chaotic dynamics, attractors, and inertial manifolds. Such transitions to turbulence occur in highly idealized laboratory experiments but rarely apply to actual flows in nature and engineering quoted in the first sentence of this review. In such flows with fully developed turbulence, the scales are so large or the flow is so rapid and the direct effects of viscosity so small that there are a tremendous number of unstable degrees of freedom, typically on the order of 10^6 to 10^{15}. Thus, an approach through finite dimensional modelling is nearly fruitless and the essentially infinite dimensional inherently statistical nature of the fluid flow must be confronted directly.

The understanding of fully developed turbulence remains perhaps the major grand unsolved problem of classical physics. Virtually all pure mathematicians have heard that the Russian probabilist Kolmogorov has contributed significantly to the theory of turbulence. In fact, in 1941, Kolmogorov predicted that the statistical features of the velocity field in the intermediate scales, the inertial range, are universal and follow the famous Kolmogorov $\frac{5}{3}$ law for the velocity spectrum. Kolmogorov deduced his law by making a few idealized plausible assumptions and cleverly applying dimensional analysis — no properties of the equations of fluid flow are utilized beyond the most elementary scaling. Nevertheless, experimental research over the last fifty years confirms many aspects of Kolmogorov's prediction. Chapter 3 of Chorin's book contains an excellent discussion and critique of Kolmogorov's theory from the modern point of view, and a recent memorial volume of the Proceedings of the Royal Society ([1]) describes some of the modern implications of Kolmogorov's work. Despite the success of Kolmogorov's theory in describing some crude features of fully developed turbulence, there are many aspects of his argument that are inconsistent with actual turbulence due to intermittency (see Chapter 3 of Chorin's book). Furthermore, it is a great theoretical challenge to develop a turbulence theory which can yield crude predictions similar to Kolmogorov's that allows for intermittency and that also is consistent with the detailed properties of actual solutions of the fluid equations.

The main goal of Chorin's book is to describe and summarize his own attempts over the last fifteen years to develop a theory which reconciles the Kolmogorov prediction with intermittency by accounting for specific features of actual solutions of the fluid equations. Chorin's main hypothesis is that the statistical stretching and folding of vortex filaments in fluid flow create the universal statistics of the inertial range. Chapters 1 and 2 of the book provide an excellent rapid introduction to the role of vorticity in fluid flow and describe some general properties of statistical fluid flows. As mentioned earlier in this review, Chapter 3 explains Kolmogorov's theory from the modern point of view, including a lucid discussion of the failings of this theory as regards intermittency. Chapter 4 is an introduction to statistical theory at large scales for two dimensional flows; the main goal of this chapter in the context of Chorin's book is to provide a more elementary pedagogical example of a statistical equilibrium theory in the context of fluid dynamics. The main hypothesis presented in the book and mentioned earlier in this review is developed in Chapters 5, 6, and 7. Here novel analogies are made between the statistics of stretching and folding of vortex lines and various powerful modern concepts in equilibrium statistical physics involving polymers, percolation, self-avoiding random walks, and the Kosterlitz-Thouless phase transition. These considerations lead to the vortex filament model and an alternative explanation of the Kolmogorov spectrum as developed by Chorin in Chapter 7.

One appealing facet of Chorin's book is the blend of modern computational results together with theory. There is a tradition extending over the last forty years of utilizing methods and analogies from quantum field theory and various branches of statistical physics in attacking problems in turbulence theory. Two other recent books on turbulence theory ([2], [3]) provide a wealth of information and a perspective on turbulence theory completely different from that developed in Chorin's monograph. Also, statistical theories for essentially two dimensional flows including the effects of rotation, stratification and topography for applications in the atmosphere and ocean are an extremely important and active research topic with many theoretical results and important implications beyond those sketched in Chapter 4 and referenced in Chorin's book; the interested reader can consult several recent articles on these topics ([4], [5], [6], [7]). Chorin's book has few rigorous mathematical proofs but instead is a unique amalgam of sophisticated mathematical ideas and tools blended together with reasoning from statistical physics and modern computing. The reader interested in a mathematically rigorous treatment of ideas in fluid flow involving vorticity and turbulence can consult the forthcoming monograph of the reviewer and Bertozzi ([8]) as well as some recent survey articles ([9], [10]).

In summary, *Vorticity and turbulence* is a wonderful book containing many novel ideas written by one of the most creative researchers of his generation in these research areas. Every student, mathematician, and researcher interested in turbulence should own a copy of this book.

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