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*Heat kernels and Dirac operators*, by N. Berline, E. Getzler and M. Vergne, Grundlehren Math. Wiss., vol. 298, Springer-Verlag, New York, 1992, vii + 369 pp., \$79.00, ISBN 3-540-53340-0

Perhaps the most famous theorem of the 1960's, the Atiyah-Singer index theorem bears all the hallmarks of great mathematics: it draws on and relates several fields in mathematics, explicating and amplifying relationships between them, and in turn contributes to the internal structure of each. Mathematicians such as Agranovic, Gelfand and Seeley had already attacked the basic problem from an analytic viewpoint, and various beguiling special cases, such as Hirzebruch's signature formula, were already known. But it was the insight of Atiyah and Singer to use K-theory to formulate and solve this problem. Another of their principal discoveries was the use of spinors and the existence of the Dirac operator in a general Riemannian context. Atiyah [At] has given an excellent account of his role in the development of the subject.

The index theorem itself equates an analytic quantity, the index of an elliptic operator $P$ on a compact manifold $M$, with certain characteristic numbers of $M$. More precisely, $P : \mathcal{C}^\infty(M;E) \to \mathcal{C}^\infty(M;F)$ is a differential (or pseudodifferential) operator carrying smooth sections of a bundle $E$ over $M$ to smooth sections of another bundle $F$. For every point of the cotangent bundle $(x,\xi) \in T^*M$ the principal symbol of $P$, $\sigma(P)(x,\xi)$, is a homomorphism carrying $E_x$, the fibre of $E$ at $x$, to $F_x$. When $P$ is differential, the symbol may be defined in local coordinates by formally replacing each derivative $\partial/\partial x_j$ by $\frac{1}{i}\xi_j$ to obtain a polynomial in $\xi$ (with coefficients depending on $x$) of order $m$, the order of $P$; the principal symbol is obtained by dropping all terms of homogeneity less than $m$. $P$ is said to be elliptic when this homomorphism is invertible for all $x \in M$ and $0 \neq \xi \in T^*_x(M)$. In particular, if $P$ is elliptic, the rank of the bundles $E$ and $F$ is the same. Basic elliptic theory shows that when $M$ is compact

$$P : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F)$$

is a Fredholm mapping: namely, its nullspace is finite dimensional, and its range is closed and has finite dimensional complement. The index of $P$ is defined to be the difference of the dimensions of the nullspace and of this complement. In terms of the adjoint operator $P^*$ it may also be expressed as

$$\operatorname{ind} P = \dim \operatorname{null}(P) - \dim \operatorname{null}(P^*).$$

The formula of Atiyah and Singer equates this index with a characteristic number of $M$ associated to a bundle over $M$ constructed using the principal symbol $\sigma(P)$.

Although it might seem unlikely that one could compute a quantity of this complexity, the index possesses considerable stability properties. In particular, if $P_t$ is a continuous one-parameter family of elliptic operators, then $\operatorname{ind}(P_t)$ is independent of $t$, even though the dimensions of the nullspace and cokernel of $P_t$ may jump as $t$ varies. The first proof of the general theorem [AS1] uses this and other, more dramatic, forms of stability (invariance under cobordism) to reduce the problem to the consideration of twisted Dirac operators on products of complex projective spaces. This proof is exposed in [Pa].

Central to this argument is the fact that these twisted Dirac operators generate, in a K-theoretic sense, and up to deformation, all elliptic operators. (This is more straightforward if the underlying manifold $M$ is spin, but somewhat subtler and less direct in general.) In making this precise, it is important to widen our scope to let $P$ vary amongst elliptic pseudodifferential operators. These are defined by replacing the symbol of $P$ by a more general function which has an expansion, as $|\xi| \to \infty$, in homogeneous functions in $\xi$ of degrees decreasing to $-\infty$. In the early '60's, pseudodifferential operators were new on the scene; only a few years earlier Seeley had demonstrated the coordinate invariance, in a suitable sense, of singular integral operators, leading to the possibility of defining them on manifolds. (Indeed, the index theorem, along with Calderon's proof of uniqueness for the Cauchy problem, was a principal motivation for the further development of pseudodifferential operator theory.)

This original proof was sufficiently complicated, and drew on enough sophisticated machinery, that there was a real motivation to simplify it. The K-theoretic part of this argument works quite generally and shows that it suffices to establish the theorem for Dirac-type operators on any manifold in order to establish the general result. Not long after, Atiyah and Singer found another argument [AS2] which proceeds by embedding the manifold into Euclidean space and applying a direct image construction from K-theory, thus obviating the use of cobordism.

Not until the end of the decade was the fundamental role played by the heat equation realized. This new sort of proof opened up many connections with physics, as well as leading to numerous generalizations of the basic theorem. McKean and Singer [McS] discovered the basic spectral cancellation. Namely, the action of $P$ yields an isomorphism between the eigenspaces corresponding to nonzero eigenvalues of the auxiliary nonnegative self-adjoint operators $P^*P$ and $PP^*$. This means that all terms arising from nonzero eigenvalues in

$$\operatorname{Tr} e^{-tP^*P} - \operatorname{Tr} e^{-tPP^*} = \sum_{j=0}^{\infty} e^{-t\lambda_j} - \sum_{k=0}^{\infty} e^{-t\mu_k} \equiv f(t)$$

cancel, where $\{\lambda_j\}$ and $\{\mu_k\}$ are the sets of eigenvalues for $P^*P$ and $PP^*$, respectively. (Other functions of $P^*P$ and $PP^*$ can (and have been) used instead; the heat kernel is a sensible choice since it can be characterized by a simple PDE, hence is more directly constructible.) Thus $f(t)$ is independent of $t$ and identically equals the index of $P$. But we may also compute its value as $t \to 0^+$. In this limit, the heat kernels may be approximated by their parametrices up to any finite order in $t$, and these are constructed from local data alone. This shows that $\operatorname{ind} P$ is the integral over the manifold of an integrand constructed purely from local data; the important problem of identifying this local integrand remains. As noted earlier, it suffices to do this for generalized Dirac operators.

The trace of the heat kernel of a second order self-adjoint elliptic operator $L$, with eigenvalues $\{\lambda_j\}$, on an $n$-dimensional manifold has an expansion of the form

$$\operatorname{Tr} e^{-tL} = \sum_{j=0}^{\infty} e^{-t\lambda_j} \sim t^{-\frac{n}{2}} \sum_{\ell=0}^{\infty} a_\ell t^\ell.$$

The coefficients $a_\ell$ are called the "heat invariants" of $L$, and they are known to be integrals of quantities depending only on a finite jet of the metric and the symbol of the operator (the order of the jet increases with $\ell$). The term of interest here is the difference in the coefficient of $t^0$ in the two heat kernel expansions.

At this stage, the problem becomes essentially algebraic. Patodi first made the calculations, by heroic brute force, for the Gauss-Bonnet operator ($d+d^*$ acting on even degree forms of a manifold), and some other natural geometric operators. He discovered "a miraculous cancellation" — namely, even though both heat invariants have quite complicated expressions, the difference between the two is much simpler and can be identified with a representative in cohomology of the appropriate characteristic class. A simpler approach was found by Gilkey [Gi] for these and other geometric operators, and the procedure was finally systematized fully using invariant theory by Atiyah, Bott and Patodi.

This final argument is still not elementary, but in the early 1980's, Getzler [Ge] discovered a very simple explanation for the miraculous cancellation for generalized Dirac operators $D$. (Closely related approaches were discovered independently, and essentially simultaneously, by the physicists Friedan, Windey and Alvarez-Gaumé.) He defined a rescaling of the underlying Clifford bundle, using the variable $t$, in terms of which the heat kernel expansions commence with $t^0$, rather than $t^{-n/2}$. (Patodi and Kasparov also understood the filtration of the Clifford bundle and its possible uses in this direction.) In fact, the heat kernel restricted to the diagonal (rather than just its trace) has an asymptotic expansion of this form. Since the leading term of an asymptotic expansion is always the easiest one to compute, it is then not hard to see that the coefficient of $t^0$ in this expansion, which is a differential form of mixed degree, has top degree component precisely the Chern-Weil representative for the characteristic class! The rescaling reveals that the correct model operator for $D^*D$ and $DD^*$ is a (quantum) harmonic oscillator, i.e. an operator of the form $\Delta + a^2|x|^2$, where the coefficient $a$ is the curvature 2-form of the metric. The form of the heat-kernel for the harmonic oscillator was discovered by Mehler in the 19th century, and inserting this curvature form into it yields the expected answer.

Identifying this local integrand (or better, the leading term in the expansion of the heat kernel on the diagonal) in more general contexts is an industry in its own right, and the result is known as the "local index theorem". Beyond its role in the proof of the original index theorem, outlined above, it is needed for the numerous generalizations of the Atiyah-Singer theorem for which the heat-kernel proof is the only one known. Chief amongst these generalizations is the Atiyah-Patodi-Singer index theorem for manifolds with boundary [APS]. Another generalization which has also proved important in physics is the families index theorem; the local index theorem in this context was originally proved by Bismut [B] using probabilistic methods, but it can also be proved quite efficiently by more direct heat-kernel methods. The local index formula of Connes and Moscovici [CM] should also be included here.

Beyond the names already mentioned, many others have contributed significantly to the development of this subject, including Quillen, Cheeger, Müller, Donnelly, Stern, Lott, Freed and Melrose, to name just a very few. The subject is still very active and expanding in many directions. To this reviewer, one of the most interesting is the extension of index theory to various classes of noncompact manifolds, including locally symmetric spaces.

The book under review, by Berline, Getzler and Vergne, is an outstanding monograph on this beautiful subject. As its title promises, the viewpoint adopted here is the systematic use of heat-kernel techniques to study a variety of topics in index theory. The topics covered reflect the tastes of the authors; many results here are due to them, and many other important results are included, often with new proofs to fit the general flow of ideas. Also noteworthy is the consistent use of the Quillen-Bismut superconnection formalism. Although in principle self-contained, this book might be a bit heavy for the utter novice, but it treats all the background material thoroughly and well. In particular, a construction of the heat-kernel for second order geometric operators is included; although this appears in other places, this is a particularly readable and complete account. The section on geometry, including the discussion of characteristic classes, is probably too brief for someone who has never encountered them before, but the discussion of superconnections is good.

There are two, essentially independent, proofs of the local index theorem given in the fourth and fifth chapters. The first of these follows the method of Getzler sketched above, while the second involves a different construction of the heat-kernel by working on the principal bundle due to Berline and Vergne. This second proof brings in the group theory quite explicitly, leading to the central focus of the next few chapters. This is an explanation of why the index density function is so similar to the Jacobian of the exponential map of a Lie group, and ultimately to the quantity appearing in Kirillov's formula for the equivariant index of a compact Lie group. Vergne's address at the Warsaw International Congress [V] gives a very nice overview of her work with Berline on these matters. Chapter 7 covers some classical material on equivariant differential forms, as well as newer results. This chapter is invaluable for the reader with a more geometric and/or analytical bent, since this material is not presented anywhere else (to my knowledge) in such a unified fashion, and other sources for it are written assuming much more knowledge of Lie theory.

The ninth and tenth chapters contain a very accessible treatment of the construction of the index bundle and the families index theory, much of which is due to Bismut.

A specific and novel goal throughout the book is the construction of *explicit* differential forms representing the index class. Although this is not strictly necessary for the index theorem itself, it becomes much more important when considering anomaly formulæ, or the construction of eta and analytic torsion forms arising in adiabatic limits.

This book joins other recent monographs on closely related topics, in particular the excellent books by Lawson and Michelsohn [LM] and Roe [R]. Roe's book is the more closely related of the two to the book by Berline, Getzler and Vergne, but it is somewhat more elementary and covers fewer topics, in particular, none of the more group-theoretic results. The book by Lawson and Michelsohn is more consistently topological and includes, for example, a more complete discussion of Clifford modules. Each of these books has a rather different viewpoint and covers rather different material than the others, but they all belong on the bookshelf of anyone seriously interested in this field. Obviously there are relevant topics which had to be left out, such as the work of Bismut, Gillet and Soulé on the arithmetic Riemann-Roch theorem, but by any standard this book is an outstanding addition to the literature.

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Rafe Mazzeo
Stanford University
*E-mail address*: mazzeo@math.stanford.edu