BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, January 1996

Sub-Hardy Hilbert spaces in the unit disk, by D. Sarason, Lecture Notes in the Mathematical Sciences, vol. 10, Wiley, New York, 1994, xiv+95 pp., \$49.95, ISBN 0-471-04897-6

Invariant subspace problems very often combine operator methods and complex function theory in a fruitful mix. In some notable cases, their study has led to extensive generalizations and applications. The present volume is a neat instance of the continuing development of invariant subspace ideas that originated a half-century ago.

Beurling's characterization of the invariant subspaces of the shift operator [4] opened the way to numerous developments in modern analysis including a new approach to interpolation theory [12]. In a related direction, a new class of Hilbert spaces was introduced by L. de Branges and the reviewer [7] in an attempt on the still open question of existence of invariant subspaces for Hilbert space operators. Beurling's theory is a particular case: the prototypical situation is an invariant subspace of the shift operator and its orthogonal complement. The new spaces have not solved the original problem of existence of invariant subspaces, but they have found other uses. In a different guise, the spaces made a startling appearance in de Branges' theory of coefficient estimates and led to the proof [5] of the Bieberbach conjecture. At about the same time, Sarason became interested in the spaces and wrote the first [13] of a series of papers developing especially connections with function theory in the unit disk. In 1989, Sarason presented a series of lectures on this topic at a conference which was held at the University of Arkansas. The present volume is a version of these lectures.

Let H^2 be the space of functions on the unit disk D which are representable as power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. We view H^2 as a subspace of L^2 on the unit circle by replacing convergent power series by their boundary functions. The Lebesgue spaces L^p are defined with respect to normalized Lebesgue measure on ∂D. Let H^∞ be the space of bounded analytic functions on D in the supremum norm, which we sometimes view as a subspace of L^∞ in a similar way. These are particular cases of the standard Hardy classes H^p for $1 \le p \le \infty$. Each φ in L^∞ induces a Toeplitz operator T_φ on H^2 defined by $T_\varphi f = P_+(\varphi f)$, where P_+ is the projection of L^2 on H^2.

The sub-Hardy spaces of the title refer to certain Hilbert spaces of holomorphic functions which are linear subspaces of H^2 but have different norms. They are induced by a function $b \in H^\infty$ of norm at most one and include the Hilbert space $\mathcal{M}(b)$ with reproducing kernel

$$\frac{b(z)\overline{b(w)}}{1 - z\bar{w}}, \tag{1}$$

and the Hilbert space $\mathcal{H}(b)$ with reproducing kernel

$$\frac{1 - b(z)\overline{b(w)}}{1 - z\bar{w}}. \tag{2}$$

These definitions show little of the nature of the spaces, however, and they are best understood through equivalent forms. We give two such:

Geometric approach.

A Hilbert space $\mathcal{G}$ is said to be **contractively contained** in a Hilbert space $\mathcal{H}$ if $\mathcal{G}$ is a linear subspace of $\mathcal{H}$ and the inclusion mapping is a contraction. In this case, we define a **complementary space** $\mathcal{F}$ as the set of vectors f in $\mathcal{H}$ such that

$$\sup_{g\in\mathcal{G}} \left[\, \|f+g\|^2_{\mathcal{H}} - \|g\|^2_{\mathcal{G}} \,\right] < \infty.$$

It can be shown that $\mathcal{F}$ is a linear space and a Hilbert space in an inner product such that the value of the supremum is $\|f\|^2_{\mathcal{F}}$. It turns out that $\mathcal{F}$ is contractively contained in $\mathcal{H}$ and the complementary space to $\mathcal{F}$ is $\mathcal{G}$. If $h = f + g$ with $f \in \mathcal{F}$ and $g \in \mathcal{G}$, then

$$\|h\|^2_{\mathcal{H}} \le \|f\|^2_{\mathcal{F}} + \|g\|^2_{\mathcal{G}}\,.$$

Every $h \in \mathcal{H}$ has a unique representation in this form for which equality holds. The inclusions of $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{H}$ are isometric if and only if the spaces intersect only in the zero element, and in this case $\mathcal{H} = \mathcal{F} \oplus \mathcal{G}$, the usual orthogonal direct sum. This notion of complementary spaces thus generalizes orthogonal complementation of closed subspaces of a Hilbert space.

Now let $\mathcal{M}(b)$ be the set of functions of the form bf with $f \in H^2$ in the norm such that multiplication by b is an isometry from H^2 onto $\mathcal{M}(b)$ (take $\mathcal{M}(b)$ to be the zero space when $b \equiv 0$). Then $\mathcal{M}(b)$ is contained contractively in H^2, and we may define $\mathcal{H}(b)$ to be the complementary space. It can be shown that $\mathcal{M}(b)$ and $\mathcal{H}(b)$ have reproducing kernels (1) and (2), so the definitions are equivalent.

Operator-theoretic approach.

If A is a contraction operator from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}$, its range $\mathcal{M}(A)$ is a Hilbert space in a unique inner product which makes A a partial isometry from $\mathcal{H}_1$ onto $\mathcal{H}$. By $\mathcal{H}(A)$ is meant the space $\mathcal{M}(B)$ where B is any operator such that $1 - AA^* = BB^*$. This definition is independent of the choice of B because two spaces $\mathcal{M}(B_1)$ and $\mathcal{M}(B_2)$ are equal isometrically if and only if $B_1B_1^* = B_2B_2^*$. In particular, we can always choose $B = (1 - AA^*)^{1/2}$.

The Toeplitz operator T_b is multiplication by b on H^2. Define $\mathcal{M}(b)$ to be $\mathcal{M}(T_b)$, and let $\mathcal{H}(b)$ be $\mathcal{H}(T_b)$. Then $\mathcal{M}(b)$ is contractively contained in H^2 and its complementary space is $\mathcal{H}(b)$. Hence the spaces are the same as before. The operator approach has been discovered independently by a number of people. The reviewer learned it in seminar lectures of M. Rosenblum in the late 1960's.

More generally, for each $\varphi \in L^\infty$ let $\mathcal{M}(\varphi)$, $\mathcal{H}(\varphi)$ be the spaces $\mathcal{M}(T_\varphi)$, $\mathcal{H}(T_\varphi)$ induced by the Toeplitz operator T_φ. The space $\mathcal{H}(\bar{b})$ corresponding to the choice $\varphi(e^{i\theta}) = \overline{b(e^{i\theta})}$ plays an important role in the theory. So do the spaces $\mathcal{M}(a)$, $\mathcal{H}(a)$ defined for the outer function a such that $a(0) > 0$ and

$$|a(e^{i\theta})|^2 + |b(e^{i\theta})|^2 = 1$$

a.e. on ∂D, whenever such a function exists. According to a classic theorem of Szegő, the function a is defined if and only if $\log\left[1 - |b(e^{i\theta})|^2\right] \in L^1$. It is known that this occurs if and only if b is not an extreme point of the unit ball in H^∞. Accordingly, we speak of **extreme point** and **nonextreme point** cases in the theory. The extreme point case includes inner functions.

The shift operator on H^2 and its adjoint are given by

$$S: f(z) \to zf(z), \qquad S^*: f(z) \to [f(z) - f(0)]/z.$$

The operator S^* leaves $\mathcal{H}(b)$ invariant, and the restriction $X = S^*|\mathcal{H}(b)$ is a contraction relative to the norm of $\mathcal{H}(b)$. The extreme point case is characterized by the fact that the identity

$$\|Xf\|^2_{\mathcal{H}(b)} = \|f\|^2_{\mathcal{H}(b)} - |f(0)|^2$$

holds for all f in $\mathcal{H}(b)$. In the nonextreme point case, it turns out that $\mathcal{H}(b)$ is also invariant under S, and then the operator $Y = S|\mathcal{H}(b)$ has a special role to play.

The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are characterized as Cauchy transforms of weighted Hardy classes. For example, let μ be the positive measure on ∂D which appears in the Herglotz-Riesz representation

$$\frac{1+b(z)}{1-b(z)} = \int_{\partial D} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(e^{i\theta}) + ic, \qquad c = \bar{c}.$$

Let $H^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$. An isometry is then defined by mapping the function $q(e^{i\theta})$ in $H^2(\mu)$ to

$$f(z) = \big(1 - b(z)\big) \int_{\partial D} \frac{q(e^{i\theta})}{1 - e^{-i\theta} z}\, d\mu(e^{i\theta}).$$

This construction can be repeated with b replaced by $\bar{\lambda} b$ for any $\lambda \in \partial D$ since $\mathcal{H}(b)$ is unchanged by this substitution; the Herglotz-Riesz measure is denoted μ_λ. The space $\mathcal{H}(\bar{b})$ has a similar characterization relative to the measure on ∂D with density $1 - |b(e^{i\theta})|^2$.

In general it is not easy to identify the functions which belong to $\mathcal{H}(b)$. There are, however, situations in which the elements of the space can be completely characterized. By a **Helson-Szegő measure** we mean an absolutely continuous measure on ∂D whose density has the form $\exp(x + \tilde{y})$ where x and y are real-valued functions in L^∞ and $\|y\|_\infty < \pi/2$ ($\tilde{y}$ is the conjugate function of y). Functions $g_1, g_2 \in H^\infty$ are called a **corona pair** if $|g_1(z)| + |g_2(z)|$ is bounded away from zero on D. A function $\varphi \in H^\infty$ is a **multiplier** of $\mathcal{H}(b)$ if φf belongs to $\mathcal{H}(b)$ for every f in $\mathcal{H}(b)$.

Theorem. *In the nonextreme point case, the following assertions are equivalent*:
1. $\mathcal{H}(b) = \mathcal{M}(a)$;
2. *for some* $\lambda \in \partial D$, μ_λ *is a Helson-Szegő measure*;
3. *for all* $\lambda \in \partial D$, μ_λ *is a Helson-Szegő measure*;
4. a, b *form a corona pair and* $T_{a/\bar{a}}$ *is invertible*;
5. *every function in* H^∞ *is a multiplier of* $\mathcal{H}(b)$;
6. Y *is similar to* S.

Various conditions imply the equality of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$.

Theorem. *In the extreme point case, the following assertions are equivalent*:
1. $\mathcal{H}(b) = \mathcal{H}(\bar{b})$;
2. b *is invertible in* H^∞;
3. X *is similar to a unitary operator.*

The nonextreme point case is developed in other directions. Then $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$ if and only if Y is quasisimilar to S. This turns out to be related to the notion of rigid functions and exposed points of the unit ball of H^1.

Other topics treated in the book include an elegant account of angular derivatives and the Denjoy-Wolff theorem by Hilbert space methods. The spectral properties of X and Y are detailed. There is a theory of multipliers, and the measures that appear in Cauchy representations are studied.

The slim size of the book belies its content. The volume is densely packed with ideas. Proofs are clear and concise. The book is well organized and easy to navigate, making it suitable for self-study and informal seminars. It should become a standard source for Hilbert space methods in function theory on the unit disk.

A comprehensive account of the subject is probably not possible, at least at this time, and there are other directions of current interest. Among omitted topics are vector and indefinite generalizations, applications to interpolation, operator colligations, and linear systems; for example, see [1–3, 6, 8–11]. Future developments will hopefully also address the parallel theory of logarithmic versions of (2) which occurs in de Branges' theory of coefficient estimates and their truncated versions. We remark that the character and goals of [7] are different, and the reviewer hopes it might continue to be consulted.

The author is to be congratulated and thanked not only on his contributions to the subject but also for putting his view forward in this timely exposition.

References

1. D. Alpay, A. Dijksma, J. Rovnyak, and H. S. V. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces,* in preparation.
2. T. Andô, *de Branges Spaces and Analytic Operator Functions*, Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo, 1990.
3. J. A. Ball and T. L. Kriete III, *Operator-valued Nevanlinna-Pick kernels and the functional models for contraction operators*, Integral Equations and Operator Theory **10** (1987), 17–61. MR **88a:**47013
4. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239-255. MR **10:**381
5. L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137-152. MR **86h:**30026
6. ______, *A construction of Krein spaces of analytic functions*, J. Funct. Anal. **98** (1991), 1–41. MR **92k:**46031
7. L. de Branges and J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York, 1966. MR **35:**5909
8. ______, *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965), 1966, pp. 295–392. MR **39:**6109
9. H. Dym, *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces, and Interpolation*, CBMS Regional Conference Series in Mathematics, vol. 71, Amer. Math. Soc., Providence, RI, 1989. MR **90g:**47003
10. J. Guyker, *The de Branges-Rovnyak model with finite-dimensional coefficients*, Trans. Amer. Math. Soc. **347** (1995), 1383–1389. MR **95g:**46045
11. N. K. Nikol′skiĭ and V. I. Vasyunin, *Notes on two function models*, The Bieberbach Conjecture (West Lafayette, Ind., 1985), Amer. Math. Soc., Providence, RI, 1986, pp. 113–141. MR **88f:**47008

12. D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203. MR **34:**8193
13. ———, *Shift-invariant subspaces from the Brangesian point of view*, The Bieberbach Conjecture (West Lafayette, Ind., 1985), Math. Surveys Monographs, vol. 21, Amer. Math. Soc., Providence, RI, 1986, pp. 153–166. MR **88d:**47014a

James Rovnyak
University of Virginia
E-mail address: rovnyak@Virginia.EDU