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Functional differential equations I: C^**-theory*, by Anatolij Antonevich and Andrei Lebedev, Pitman Monographs & Surveys in Pure & Appl. Math., vol. 70, Longman Scientific & Technical, Harlow, Essex, England, 1994, 504 pp., \$195.00, ISBN 0-582-07251-4

The theory of Banach algebras was inspired by problems related to solving linear equations in infinite dimensional spaces. The abstract theory that evolved, in turn, has proven over and over again to be extremely helpful in the concrete analysis of such problems. Many equations of interest, particularly those that involve nonlocal functional differential operators, can be expressed in terms of linear combinations of "shifts of operators" with coefficients coming from a given Banach algebra. These lead naturally to larger Banach algebras obtained as "crossed products" of the Banach algebra of coefficients by the groups generated by the shifts. This is the subject of the book under review.

To be a bit more specific and for illustrative purposes, consider the simple-looking, convolution-like equation:

$$f(x) + \int_{-\infty}^{\infty} k_1(x-y)f(y)dy + e^{ix\alpha}\int_{-\infty}^{\infty} k_2(x-y)f(y)dy + e^{ix\beta} \times \int_{-\infty}^{\infty} k_3(x-y)f(y)dy = g(x).$$

The function g is prescribed, say, in $L^2(\mathbf{R})$, and the function f, also in $L^2(\mathbf{R})$, is sought. The functions, or *kernels*, k_i, $i = 1-3$, are assumed to lie in $L^1(\mathbf{R})$, a Banach algebra under convolution. In terms of Fourier transforms this equation may be re-expressed as

$$\hat{f}(\lambda) + \hat{k}_1(\lambda)\hat{f}(\lambda) + \hat{k}_2(\lambda-\alpha)\hat{f}(\lambda-\alpha) + \hat{k}_3(\lambda-\beta)\hat{f}(\lambda-\beta) = \hat{g}(\lambda).$$

If $\alpha = \beta = 0$, then one may combine all the kernels into one to obtain the equation:

$$\hat{f}(\lambda) + \hat{k}(\lambda)\hat{f}(\lambda) = \hat{g}(\lambda),$$

which can always be solved for $f \in L^2(\mathbf{R})$, provided $\hat{k}(\lambda) \neq -1$ for all λ. Just divide both sides by $1+\hat{k}$, and take inverse Fourier transforms. This, of course, is well known and quite classical. When either α or β is not zero, the situation is more complicated, and Fourier transform methods alone do not suffice.

It is convenient, for most purposes, to pass from the original Banach algebra $L^1(\mathbf{R})$ to the slightly larger and more manageable commutative Banach algebras, $C_0(\mathbf{R})$, the continuous complex valued functions on $\mathbf{R}$ that vanish at infinity, and the algebra one obtains from $C_0(\mathbf{R})$ by adjoining the identity, $C(S^1)$, the continuous functions on the one-point compactification of $\mathbf{R}$. The passage is made via the Fourier transform, and one really ought to distinguish between the first use of $\mathbf{R}$, which supports the original functions, f, g, k_i, and the second, which supports their Fourier transforms. However, as is customary, we shall not distinguish notationally between the two uses. The advantage of $C_0(\mathbf{R})$ and $C(S^1)$ over $L^1(\mathbf{R})$ is the fact that they are C^*-algebras, meaning that they come equipped with an involution $*$ (complex conjugation in this case) which satisfies the equation $\|f^*f\| = \|f\|^2$.

This makes for important simplifications in the theory, but they need not concern us here.

What is important for this discussion is the fact that the Fourier transform converts the original equation into one which involves coefficients coming from the C^*-algebra $C_0(\mathbf{R})$, viewed as multiplication operators on the Hilbert space $L^2(\mathbf{R})$, and the translation operators determined by α and β, U_α, and U_β, where $U_\alpha\xi(\lambda) = \xi(\lambda - \alpha)$ and similarly for U_β. Thus the natural place to study the solvability of the original equation is in the C^*-algebra $\mathcal{A}$ on $L^2(\mathbf{R})$ generated by $C_0(\mathbf{R})$ and these translations. That is, $\mathcal{A}$ is the norm-closed, selfadjoint algebra of bounded operators on $L^2(\mathbf{R})$ generated by $C_0(\mathbf{R})$ and the operators U_α and U_β. This algebra, however, can be quite subtle. For example, if α or β (but not both) is zero or if their quotient is *rational*, then $\mathcal{A}$ is isomorphic to the C^*-algebra of all compact operators on Hilbert space tensored with $C(S^1)$. While it is not necessary to know this in order to solve the original equation, it can be helpful, and the book under review makes this clear. If α/β is an *irrational* number, then $\mathcal{A}$ is simple, and an understanding of the techniques described in this book should prove invaluable for analyzing the equation.

For the general setup considered in the book, suppose that A is a C^*-algebra and that G is a (discrete) group acting on A by $*$-automorphisms; then the *crossed product* of A by G is the C^*-algebra, denoted $A \times G$, obtained as follows. First, let $C_{cc}(G, A)$ denote all the A-valued functions on G that vanish outside a finite set (depending on the function). Under pointwise addition and scalar multiplication, this space becomes a $*$-algebra with product

$$f * g(t) = \sum_{s\in G} f(s)\sigma_s(g(s^{-1}t))$$

and involution

$$f^*(t) = \sigma_t(f(t^{-1})^*),$$

where σ_s denotes the action of $s \in G$ on A. A *covariant representation* of $C_{cc}(G, A)$ on a Hilbert space $\mathcal{H}$ is a pair (U, π) where U is a unitary representation of G on $\mathcal{H}$ and π is a $*$-representation of A on $\mathcal{H}$ such that

$$\pi(\sigma_s(a)) = U(s)\pi(a)U(s^{-1})$$

for all $a \in A$ and all $s \in G$. Such a pair enables one to define a representation of $C_{cc}(G, A)$ on $\mathcal{H}$, denoted $U \times \pi$, through the formula

$$U \times \pi(f) := \sum_{s\in G} \pi(f(s))U(s).$$

The C^*-algebra $A \times G$, then, is defined to be the completion of $C_{cc}(G, A)$ in the norm

$$\|f\| = \sup \|U \times \pi(f)\|$$

where the supremum is taken over all covariant representations of $C_{cc}(G, A)$. (This supremum is finite and is dominated by $\sum \|f(s)\|$.)

Complicated as the definition might look, crossed products prove to be very useful in a variety of situations, and the definition puts into evidence the key ingredient for their analysis, viz. covariant representations. Returning to our original equation, it is easy to see, after taking Fourier transforms, that a crossed product that can be associated with it is $A \times G$, where $A = C_0(\mathbf{R})$, and where $G = \mathbf{Z}^2$ acts on A via the formula

$$\sigma_{(n,m)}(f)(x) = f(x + n\alpha + m\beta)$$

$f \in A$. The covariant representation evident from the equation is (U, π) where $U(n,m)\xi(x) = \xi(x + n\alpha + m\beta)$, $\xi \in L^2(\mathbf{R})$, and where for $f \in A$, $\pi(f)$ is multiplication by f on $L^2(\mathbf{R})$. It is then clear that $\mathcal{A} = U \times \pi(A \times G)$. Furthermore, the fact that many crossed products are constructed out of simple objects in explicit ways enables one to calculate effectively their ideal structures. Indeed, there is a fairly large literature devoted to this problem. The game plan for analyzing equations like the original equation, then, is this: Given the equation, identify a crossed product $A \times G$ which models it. Then find the covariant representation $U \times \pi$ which enables one to pass from $A \times G$ to the specific equation. Identify the kernel of $U \times \pi$, and then identify the ideal, if any, in the quotient to which the operator giving rise to the equation belongs. If there is none, then the original equation is solvable. The book under review is about how to implement this plan.[1]

The authors intend this volume to lay the abstract ground work for a second volume devoted to a detailed analysis of functional differential equations. It begins with an enticing introductory section that describes a number of interesting examples. Presumably, these will be the subject to the second volume. It is then followed by a long, preparatory section of statements of miscellaneous results from a variety of disciplines that will be called upon throughout the sequel. The prototypical operators for all that are considered in the book, weighted shifts (including weighted translation operators), are analyzed in detail in Chapter 1. The emphasis is on the location of the spectra of such operators. The second chapter is devoted to the theory of discrete crossed products and, in particular, to conditions under which certain covariant representations of them are faithful. The third, and longest chapter, is about the interplay between the previous two: the spectral analysis of weighted shift operators arising from covariant representations of crossed products. The final chapter is devoted to questions of localization: the information that can be gleaned about the spectrum of an operator by passing to ideals in algebras that contain it.

The book is about a subject that appears to have a lot of potential. However, as an introduction and as a survey of what is known, it has numerous shortcomings. First of all, the presentation is rather uneven. Many elementary topics are belabored unnecessarily, while advanced topics are sometimes approached almost whimsically. This does not appear to be a consequence of the fact that distinct parts of the book were written by the authors separately. It is very difficult to decide for whom the book is best suited.

The book is about spectral theory and the interplay between it and algebra. Spectra are computed and Fredholm properties are sometimes analyzed, but rarely are indices computed, nor is there any mention of K-theory. Index theory and K-theory are intimately tied to the kind of analysis the authors want to perform. It is unfortunate that nothing of this very large and important subject is mentioned.

The bibliography is not as complete as it could be or, I believe, should be. For example, I find it unsettling that the primary reference to matters about crossed

[1]Technically speaking, this description of the game plan is incomplete, but the technicalities are not important for this discussion.

products is Pedersen's treatise [1]. His is an excellent exposition for a lot of purposes; but since the only crossed products considered by the authors involve discrete groups, it is odd that Zeller-Meyer's pioneering work [2] is never mentioned. In [2], one finds many of the results about the crossed products that the authors use, presented in a fashion that is much easier for the uninitiated to grasp than that found in [1]. Many of the references are, of course, written in Russian, but no citations of English translations are given. Much of the literature, even the Russian literature, that could be cited is not, but many ancillary, unnecessary items are included in the bibliography.

The presentation is marred by numerous misprints and grammatical errors. Furthermore, the printing is a photo-reproduction of an antiquated technical word processing product. These drawbacks make the book physically difficult to read.

This book raises some very serious issues that must be faced by the mathematical community. I am confident that the book faithfully reproduces the manuscript that the authors presented to the publishers. It is clear that the publishers provided no editorial help, and certainly there was no typesetting provided. I find it difficult to believe that the publishers had the manuscript critically reviewed. While there are positive things to be said for it, the book as it stands is uneven and, in essential ways, unfinished. It is the sort of thing one might publish in Pitman's Research Notes, but not in their flagship series, Surveys in Pure and Applied Mathematics. Yet it is published in this series and at the hefty price of $195. My frustrations are with the publishers, not the authors. The publishers evidently gave the authors no service and are now exploiting the universal practice followed by libraries of maintaining standing orders. Given the current declining state of libraries around the world, the community must become more critical about how the diminishing resources devoted to acquisitions are allocated. One must seriously ask, "Is this book worth the price?" I have told my librarian, "No."

References

[1] G. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London and New York, 1979. MR **81e:**46037

[2] G. Zeller-Meyer, *Produits croisés d'une C^*-algèbre par un groupe d'automorphismes*, J. Math. Pure. Appl. **47** (1968), 101–239.

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