BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, April 1996

Mutations of alternative algebras, by Alberto Elduque and Hyo C. Myung, Kluwer Academic Publishers, Boston, 1994, xiii + 226 pp., \$99.00, ISBN 0-7923-2735-7

At the core of nonassociative algebra theory are found the 8-dimensional algebra of octonions and certain algebras obtained from associative algebras by modifying the product. The most important examples of the latter are the Lie algebras which arise by using the Lie product $[x, y] = xy - yx$ on an associative algebra. Using the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ on an associative algebra gives a Jordan algebra.

It is aesthetically more pleasing, as well advantageous from a theoretical point of view, to study a class of algebras which is closed under taking subalgebras or homomorphic images, and under forming infinite direct products. Such a class of algebras is a variety, i.e., the class of all algebras satisfying some set of identities. The variety of Lie algebras is defined by two identities, $[x, y] = -[y, x]$ (anticommutativity) and $J(x, y, z) := [[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (the Jacobi identity), and it is exactly the class of all algebras which arise as subalgebras of associative algebras under the Lie product. By contrast, the class of all algebras which arise as subalgebras of associative algebras under the Jordan product (the class of special Jordan algebras) is not a variety, since it is not closed under taking homomorphic images. These algebras are contained in the variety of Jordan algebras which is defined by the identities $x \circ y = y \circ x$ (commutativity) and $((x \circ y) \circ x) \circ x = (x \circ y) \circ (x \circ x)$ (the Jordan identity). The variety of Jordan algebras also includes the Albert algebras which are forms of the algebra of 3×3 Hermitian matrices over the octonions and which are not special Jordan algebras. For characteristic not 2, any simple Jordan algebra is either special or is an Albert algebra.

The natural variety in which to place the octonions (or Cayley-Dickson algebras) is the variety of alternative algebras which is defined by the identities $x^2y = x(xy)$ and $(yx)x = yx^2$ and which includes all associative algebras. Any simple alternative algebra is either associative or is a form of the octonions. Taking an alternative algebra under the Lie product $[x, y] = xy - yx$ leads to a Malcev algebra. The variety of Malcev algebras is defined by anticommutativity and by $J(x, y, [x, z]) = [J(x, y, z), x]$ (the Malcev identity). Any simple Malcev algebra is either a Lie algebra or is a subalgebra of dimension 7 of a form of the octonions under the Lie product.

A natural generalization of the variety of Jordan algebras is the variety of noncommutative Jordan algebras which is defined by the Jordan identity and $(x \circ y) \circ x = x \circ (y \circ x)$ (flexibility). In the classification of simple finite-dimensional noncommutative Jordan algebras, a new class of algebras shows up, namely, the quasi-associative algebras. An algebra B is called quasi-associative if, after possibly making a scalar extension of degree 2, B rises up to isomorphism from an associative algebra A by using the new product $x * y = \lambda xy + (1 - \lambda)yx$ where λ is in the underlying field. Many other varieties of nonassociative algebras have been studied; and the results can be summed up by saying that when the identities of a variety are strong enough to classify the simple algebras in the variety, they always seem to be assocative, Lie, Jordan, quasi-associative, Malcev, or the octonions. With the exception of the Albert algebras, each of these algebras is alternative or else is derived from an alternative algebra by altering the multiplication (followed by possibly taking a subalgebra). Thus, associative algebras and the octonions with altered multiplication play a central role in nonassociative theory.

These altered multiplication constructions are simultaneously generalized by the concept of a mutation of an algebra. If A is an associative algebra and if $p, q \in A$ are fixed elements, we define the (p, q) mutation of A, denoted by A_{pq}, to be the vector space of A under the new multiplication $x*y = xpy - yqx$. This notion actually first arose in physics, in a quantum mechanical version of classical Hamiltonian mechanics. It was then pursued further by various nonassociative algebraists, culminating in the book under review. In fact the authors have generalized to mutations over alternative algebras, which means in practice that many proofs are divided into the case when A is associative and the case when A is the octonions. Whereas most papers dealing with mutations have assumed that A was finite dimensional, this book gives a thorough treatment of the case when A is an Artinian alternative algebra. One advantage of the approach of this book is that the reader gets a good feeling for how the Lie product, Jordan product, and quasi-associative product fit into a larger picture.

The authors have given a very careful and thorough treatment of their subject starting almost from scratch. They cover, and in many cases prove, just about any fact that might be relevant to their topic (including virtually everything mentioned in this review). They explain at the beginning of the book and at the beginning of each chapter what will be covered where, and they give ample references to the literature, either to tell the reader where to find a proof or result that is too peripheral to include or to say from what source they have taken a proof. Many false conjectures which the reader might make are quickly disposed of with counterexamples. Anyone with a little knowledge of modern algebra who wants to learn about mutation algebras will find this book an easy and enjoyable way to learn. It covers anything that one might want to know about mutation algebras as well as many related topics.

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