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Harmonic analysis techniques for second order elliptic boundary value problems, by Carlos E. Kenig, CBMS Regional Conf. Series in Math., no. 83, Amer. Math. Soc., Providence, RI, 1994, xii + 146 pp., \$30.00, ISBN 0-8218-0309-3

The last twenty years has seen considerable and fruitful research in the field of nonsmooth boundary value problems (BVP's) for partial differential equations. The objective is to understand the behavior and properties of solutions to either variable coefficient equations with minimal regularity assumptions on the coefficients or to linear constant coefficient equations in domains with nonsmooth boundary. An example of the latter is the following. Let $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ be the Laplacian in $\mathbf{R}^n$, and let $\Omega \subset \mathbf{R}^n$ be a connected bounded domain whose boundary may have corners and edges. One may then pose the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

and the object is to define and uniquely solve this problem for a broad class of data f. An example of the former, whose study forms a significant portion of the book under review, is the following. One considers a second order linear partial differential operator of the form $L = \sum_{i,j} D_i a_{ij}(x) D_j$ (which we denote hereafter as $L = \operatorname{div} A\nabla$) where $A = (a_{ij}(x))$ is a matrix whose real coefficients satisfy minimal smoothness properties, together with an ellipticity condition. Ellipticity is a strong positive definiteness of the matrix A: there exists $\lambda > 0$, so that for all x and all $\zeta \in \mathbf{R}^n$, $\lambda^{-1}|\zeta|^2 \leq \sum_{i,j} a_{ij}(x)\zeta_i\zeta_j \leq \lambda|\zeta|^2$. One then seeks to define, in some sense, Dirichlet conditions on the boundary of the domain and uniquely to solve this Dirichlet problem for a broad class of data.

There are of course connections between these two problems. Consider the Laplacian in the upper half space $\mathbf{R}^2_+$, and let $\Phi\colon \mathbf{R}^2_+ \to \mathbf{R}^2_+$ be a quasiconformal change of variables. Let L be the pullback of Δ under Φ. Then L is an elliptic operator of the form $\operatorname{div} A\nabla$, but the coefficients of A are merely bounded and measurable [CFK]. (These changes of variable give rise to an important, and computable, class of examples of this theory.) Or consider Laplace's equation in $\Omega = \{(x,t) \in \mathbf{R}^n \times \mathbf{R}_+ : t > \phi(x)\}$, the unbounded domain above the graph of a Lipschitz continuous function ϕ. A "flattening" of the domain by means of the change of variables $(x,t) \to (x, t - \phi(x))$ transforms Δ into a divergence form elliptic operator whose coefficients depend on the derivatives of ϕ and are hence at best bounded and measurable. In fact, this transformation yields an equation which is maximally smooth in a transverse direction, since the coefficients are independent of the t variable. It turns out that one can solve many natural BVP's for these "time-independent" second order operators [JK]. This last observation is then the foundation for a theory of elliptic divergence form operators under some minimal smoothness conditions in the transverse direction. These conditions, imposed on the coefficients, have an origin in the theory of differentiation as well as rich connections to modern ideas in harmonic analysis.

Let us turn then to a discussion of the origin and some motivation for these studies. The theory of second order divergence form elliptic equations described above began with the results of De Giorgi [De], Nash [Na] (for parabolic equations as well), and Moser [Mo] in the 1950s. In the first place, what can we mean by a solution u to $\operatorname{div} A\nabla u = 0$ when the coefficients $a_{ij}(x)$ are not even differentiable? We say that $\operatorname{div} A\nabla u = 0$ in a domain $\Omega \subset \mathbf{R}^n$ when $\int_\Omega A(X)\nabla u(X).\nabla\phi(X)\,dX = 0$ for every smooth function ϕ compactly supported in Ω. Here ∇u denotes the weak derivative of u, so that weak solutions are a priori defined only almost everywhere with respect to Lebesgue measure. Remarkably, the results of De Giorgi, Nash, and Moser show that weak solutions are in fact Hölder continuous of some order depending on the ellipticity constant of L, even when the $a_{ij}(x)$ are merely bounded and measurable. It is therefore possible to speak of the pointwise values of these solutions and to ask for continuous solutions of the classical Dirichlet problem ($Lu = 0$ in $\Omega, u|_{\partial\Omega} = f$ with f continuous).

The motivation for considering equations with merely bounded coefficients is twofold. First, one may be able to regard a given nonlinear PDE as a linear elliptic equation—one whose coefficients now depend on the solution of the original nonlinear problem. If a solution exists which is known to be bounded, then this divergence form operator has bounded measurable coefficients. The improvement of regularity result implies that the solution is in fact Hölder continuous. Now the coefficients of the linearized equation have improved smoothness which, in turn, implies greater regularity of the solution, and so on. When De Giorgi first proved this regularity result, a number of important nonlinear problems were solved. More recently, this theory of elliptic equations with minimal smoothness assumptions has been used in a crucial way by Caffarelli et al. for free boundary problems [AgCS, AC, C1, C2]. Another fundamental reason for the recent development and interest in this field lies in the dilation invariant nature of the problems and the rich theory that has ensued when certain powerful techniques from harmonic analysis are applied. Indeed, the PDE problems themselves have generated new methods and ideas and have contributed to fundamental developments in harmonic analysis. It is exactly this exciting interplay of ideas which is the subject proper of Kenig's book, based on his series of CBMS lectures in 1991.

The motivation for pushing the classical theory of solutions to operators like the Laplacian from smooth domains to more general domains is very similar. Recall that the classical Dirichlet problem (continuous data, with solutions continuous up to the boundary) is solvable for Δ in any domain which satisfies an exterior cone condition [Z]. Lipschitz domains are those which satisfy, uniformly, both an interior and exterior cone condition. The linear equations under consideration here are dilation invariant, and so is the class of Lipschitz domains; one seeks to understand the behavior of solutions u in Ω by means of estimates which relate the size (loosely speaking) of u to that of its boundary values, where the constant in such an inequality depends on the Lipschitz "character" of Ω. This constant is hence invariant under dilations of the domain. Thus the theory is a study of this interaction between dilation invariance of domains and equations in the presence of homogeneity and ellipticity conditions. In addition, Lipschitz domains already appear even in the classical smooth theory via the definitions of nontangential limits of harmonic functions; these domains also arise quite naturally in terms of the minimal smoothness needed for the existence almost everywhere of a normal vector to the boundary and for the validity of certain trace and extension theorems for Sobolev spaces.Thus one can consider a variety of natural boundary conditions for these elliptic equations. (Other types of equations as well, parabolic for instance, have been and are the object of research–see [B] for example.)

In order to be more precise about the aforementioned connections to harmonic analysis, let us define our boundary data as well as a technical concept which will capture the precise sense in which a solution to L in a domain Ω converges to this boundary data. Given a Lipschitz domain Ω, one associates to each $Q \in \partial\Omega$ a (truncated) cone $\Gamma(Q)$ compactly contained in the interior of Ω. If $u(X)$ is defined in Ω, the nontangential maximal function of u is $u^*(Q) = \sup\{u(X) \colon X \in \Gamma(Q)\}$. One particular aim is to solve the Dirichlet problem and to be able to prescribe boundary data in various Banach spaces, for example, in $L^p(\partial\Omega)$ with respect to surface measure on the boundary. Specifically, if f is a given function in $L^p(\partial\Omega)$, then one seeks a unique solution to $Lu = 0$ in Ω such that u converges nontangentially to f and u^* belongs to $L^p(\partial\Omega)$ together with the estimate $\|u^*\|_{L^p} \leq C\|f\|_{L^p}$ (which implies uniqueness).

First of all, why is it interesting to consider data in L^p? The case $p = \infty$ of the Dirichlet problem for these second order equations is just a weaker version of the maximum principle. The case $p = 2$ is also a natural class of data in view of the Hilbert space structure and the variational methods which come into play. In fact it turns out to be important to consider solvability questions in the full range of p's: it yields refined information about the behavior of harmonic measure (defined later), and also, for higher order elliptic equations, there are strong connections between L^p boundary value problems and the validity of maximum principles of Agmon-Miranda type [PV1, PV2].

It is not always possible to solve such a boundary value problem. The question is, under what conditions on L or Ω and for what values of p is this L^p Dirichlet problem solvable? The book under review is an ambitious and successful attempt to give a fairly detailed account of what is known about this question. Indeed, the scope of the book is broader than this, as it provides an account (sometimes in the form of remarks and notes on further research) of related developments in the theory of parabolic equations, second order elliptic equations with complex coefficients, nondivergence form equations, systems of equations and equations of higher order, as well as focusing on other BVP's such as the Neumann and regularity problems. Because the author has been instrumentally involved in all of these developments, in the formation both of key ideas and of the techniques through which these ideas are realized, the perspective presented here is unique in its breadth and clarity.

The book begins with the De Giorgi-Nash-Moser theory (some proofs are omitted), the definition of weak solutions, properties of Green's functions, and properties of harmonic measure. The existence of this "harmonic" measure on the boundary of a domain associated with an operator $L = \operatorname{div} A\nabla$ is one of the fundamental reasons that techniques of harmonic analysis play such an important role. This measure is perhaps familiar to readers from the study of analytic functions in the plane. Suppose u is a harmonic function, i.e., a solution to $\Delta u = 0$ in a domain $\Omega \in \mathbf{R}^n$. Then there exists, for each $X \in \Omega$, a representing measure ω^X such that if u is the harmonic function with boundary values f, then $u(X) = \int_{\partial\Omega} f(Q)\, d\omega^X(Q)$. The $\{\omega^X\}$ form a mutually absolutely continuous family of probability measures, whose existence is derived from the maximum principle for solutions with continuous data f by the Riesz Representation Theorem. Because the measures are mutually absolutely continuous, one generally singles out one point $X_0 \in \Omega$ and refers to $d\omega = d\omega^{X_0}$ as the harmonic measure for Δ relative to Ω. The properties of this measure, which are determined by the geometry of the domain Ω, translate into properties of solutions u and in particular whether estimates of the form $\|u^*\|_p \leq C\|f\|_p$ are valid and for which p's. Likewise, associated to an elliptic operator $L = \operatorname{div} A\nabla$ there is a family $\{\omega_L^X\}_{X\in\Omega}$ of mutually absolutely continuous probability measures so that a solution to $Lu = 0$ with $u = f$ on $\partial\Omega$ is represented by $u(X) = \int_{\partial\Omega} f(Q)\, d\omega_L^X(Q)$. For some $X_0 \in \Omega$, the measure $d\omega_L^{X_0}$ is referred to as the elliptic or harmonic measure associated to L in Ω. One connection between harmonic analysis and properties of solutions to L comes from investigating the relationship between $d\omega$ and $d\sigma$ (Lebesgue measure on $\partial\Omega$) via the general theory of weights, initiated in [M] and [CF].

Historically, the next major development in this field was the paper by Hunt and Wheeden [HW] on estimates for harmonic measure. This was followed by Dahlberg's proof that, for domains with Lipschitz boundary, the estimate $\|u^*\|_{L^2} \leq C\|f\|_{L^2}$ is valid for harmonic functions in Ω with boundary values $f \in L^2$ [D1]. The "2" is a sharp lower bound in the sense that for $p < 2$ one can exhibit a Lipschitz domain where uniqueness fails for solutions to Laplace's equation for the L^p Dirichlet problem. A result of this type is not merely a technical one: it yields a deep understanding of the role played by the geometry of this class of domains. It shows how certain precise local behavior of solutions, like the behavior of a harmonic function near an isolated conical point on the boundary, extends to all Lipschitz domains—even though such domains may contain a dense set of such singularities.

Kenig's book follows a different path, choosing instead to present first the more modern results and definitions for general divergence form elliptic operators. What follows then is a discussion of the three main problems associated with these second order operators: Dirichlet, Neumann, and regularity. Then the Beurling-Ahlfors theory of quasi-conformal mappings of the upper half plane is introduced, together with some of its implications for the elliptic theory. In the following chapter we see a return to the theory of harmonic functions and to the development of the method of layer potentials.

It is the success of the method of layer potentials for solving these BVP's which furnishes the next historical development and connection with harmonic analysis. And for this success, one requires the theory of singular integral operators (SIO's). The method of layer potentials is an approach to solvability of BVP's for, say, Laplace's equation, by means of integral equations. Here is a very brief sketch of the method. Let $\Gamma = c_n|X - Y|^{2-n}$ be the fundamental solution of the Laplacian in all of $\mathbf{R}^n$. One attempts to solve the Dirichlet problem by representing the solution as the integral over the boundary of the normal derivative of Γ against an unknown function h. The boundary data of a solution thus represented is given, in the limit, by an operator acting on h of the form $\operatorname{Id} + K$. Thus to solve this problem, in a domain $\Omega \subset \mathbf{R}^n$, one needs to invert $\operatorname{Id} + K$ on a space of functions defined on $\partial\Omega$. If the boundary of the domain Ω is sufficiently smooth (C^1 in fact), the above operator K is compact and the Fredhom theory applies [Ca, FJR]. But when Ω is merely Lipschitz, K is not compact. The operator K is an example of a SIO, and such operators, arising in PDE, were a major source of motivation for the study of SIO's initiated by Calderon and Zygmund and subsequently developed by E. Stein and many others. (Regretfully, this review neglects, due to limitations of space, the mention of many major contributors to these and related fields. Happily, the book does not, citing almost 250 sources and giving detailed accounts of many contributions.)

Before answering the invertibility question for $\mathrm{Id} + K$, one must first establish that K is a bounded operator on some space, say, L^p. In the case of Lipschitz domains this is a nontrivial issue and one whose answer awaited the work of Calderon [Ca] in the small Lipschitz constant case and [CMcM] in 1981 in the general case. In this latter work, Coifman, McIntosh, and Meyer proved that the Cauchy integral, defined on a domain with Lipschitz boundary, is a bounded operator on $L^p(\partial\Omega, \partial\sigma)$ for $1 < p < \infty$. This theorem paved the way for the application of layer potential methods to this class of domains. In 1984, Verchota [V] understood how to invert the operator $\mathrm{Id} + K$, which is the real part of the Cauchy integral in dimension 2, thereby settling a long-standing question whose positive resolution was not entirely expected.

In his proof of invertibility, Verchota realized how to use a certain Rellich-type identity which had been rediscovered a couple of years earlier by Jerison and Kenig. (It is an equation which results in a relationship between the normal and tangential derivatives of harmonic functions.) This identity, and their observation in [JK] of its utility in solving BVP's for second order divergence form elliptic operators, has an importance for this field which cannot be overstated. It is fair to say that, after [JK], most of this successful work—even in related theories, like parabolic equations, systems of equations, and higher order equations—has involved a refinement of or adaptation of the idea behind this identity. Of course the subsequent work involves many new ideas as well, but the results and observations in [JK] opened a door which changed the field.

The Rellich identity was used by Jerison and Kenig to solve not only the Dirichlet problem for certain variable coefficient operators but also the Neumann and regularity problems as well, for data in L^2. Verchota's solution of the Dirichlet problem for Laplace's equation by the method of layer potentials extended also to the regularity problem (and here for the optimal range of data). Later, Dahlberg and Kenig [DK] showed how to solve the Neumann problem in this optimal range as well by defining and solving a BVP in the atomic Hardy space. The use of the Hardy space and its dual, the space BMO [JN], provides yet another serious connection with harmonic analysis. These results and the use of Hardy spaces in related developments in the field are explained early in Chapter 2.

The latter portion of this second chapter is devoted to the perturbation theory of divergence form elliptic operators, initiated in [FJK] and developed in [D2], [Fe], [FeKP], [KP1], [KP2] and [L]. This includes the approach of multilinear singular integrals, which has the flexibility of applying to the situation of complex coefficients also. In fact the most important applications are to complex valued operators. Originally, the method of multilinear SI's was developed in connection with parabolic equations in [Fa] and [FSW], but its most notable use [CMcM] is in the elliptic theory and in its ramifications for the Kato square root conjecture. (See also [CDM].) Unfortunately the method has yielded only partial results in higher dimensions.

A more complete set of results for perturbations of elliptic operators is available in the real-valued setting. Here, the question is essentially the following. Suppose one is given an operator $L_0 = \mathrm{div}\, A_0(X)\nabla$ defined in some domain Ω, where as usual A_0 is an elliptic matrix of bounded coefficients, for which one has a priori solvability of some BVP (Dirichlet or Neumann for example) with data in L^p along with the estimates on the nontangential maximal function of u. Let $L_1 = \operatorname{div} A_1(X)\nabla$ be another such operator where $A_1(X)$ is perturbation of $A_0(X)$ in the sense that $A_0 = A_1$ on the boundary of Ω. Then, inside Ω, how close should A_1 be to A_0; i.e., what quantitative condition on $|A_1(X) - A_0(X)|$ suffices, to conclude solvability of the same BVP for L_1 with data in some L^q space?

The motivation for conditions that are sufficient, and in some cases necessary, comes from a striking analogy with results in the classical theory of differentiation. For example, consider a function $f\colon \mathbf{R}^n \to \mathbf{R}$ which verifies the Zygmund type condition $|f(x+t) + f(x-t) - 2f(x)| = |t|\eta(|t|)$ where η is an increasing function satisfying $\eta(0) = 0$. The question of classical differentiation theory is, what condition on η guarantees that f be differentiable almost everywhere? The answer, provided by Calderon and Zygmund in [CZ] (but see also [WZ] and [JN]) is that η must verify a square Dini condition: $\int_0^\infty \eta^2(s)s^{-1}\,ds < \infty$. It was first realized in [FJK] that this quantitive expression plays an important role in the theory of perturbations of operators. That is to say, the perturbation condition one needs to impose on $|A_1(X) - A_0(X)|$ has a very similar character. A rather complete understanding of these conditions, including sharpness results, was given in [FeKP], where further connections with the theory of weights were established.

The book contains, in section 6 of Chapter 2, a beautiful exposition of these ideas, developing the historical perspective within the context of these contemporary results. It is rather densely written in this section; each sentence contains important information. But Kenig draws out the beautiful links between all these ideas with such insight that the reader who spends the necessary time to absorb it is richly rewarded.

Let us conclude with some additional description of the structure and features of the book. Much of the background material is presented, not explained; and while many details of the more recent theorems are given, it is often the case that technical lemmas are merely sketched. Had all the details been presented, this monograph would have easily been three or four times as long. I think the author's judgment was to give exactly enough of the necessary details to provide the heart of the matter to experts in the field, as well as to give the overall flavor of the subject to nonexperts in analysis. It is a shame, however, that the nonlinear applications were not discussed, not even in the form of remarks in the "Notes" sections.

As mentioned earlier, the references are dizzyingly complete, and the "Notes" sections at the end of the chapters pinpoint these sources and place them in the proper historical context. Indeed, the "Notes" sections are much more than a series of references and acknowledgements. Here one finds the author's sense of the development of the subject. It is an interesting, if condensed, glimpse of a great deal of analysis in the last forty or fifty years. Less condensed, and truly inspired, is the third and final chapter on further results and open problems. It is not merely a list of problems (which is important in and of itself), but a view of the development and direction that this field can and should take. It is easy enough to make a list of difficult open problems which have resisted the best efforts to date. And a few of those problems are included here. But it is not easy and it is much more valuable to do what Kenig has achieved. This chapter contains a collection of ideas and tractable problems that arise from a real knowledge of the field and a clear vision of how to make progress, in both reasonable and significant ways. The scope of this final chapter is impressive.

This book could be used in a course for advanced graduate students, supplemented by some of the references for the earlier material. It is well written and already organized in the form of a course. The monograph will also be an excellent source book for amateurs as well as experts in this subject. And finally, it is inexpensive. The AMS, in publishing this series, has done the mathematical community a real service by providing timely and scholarly research manuscripts at such a reasonable price.

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