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Spectral theory of approximation methods for convolution equations, by Roland Hagen, Steffen Roch, and Bernd Silbermann, Operator Theory: Advances and Applications, vol. 74, Birkhäuser Verlag, Basel, Boston, and Berlin, 1995, xii + 373 pp., $124.00, ISBN 3-7643-5112-8

The mathematician's job includes solving equations $Ax = y$ with concrete linear bounded operators A on concrete infinite-dimensional Banach spaces X. Such equations can be solved explicitly in rare cases only, which motivates the search for appropriate approximation methods. In this connection, several projection methods have been enjoying great popularity for many decades.

Solving the equation $Ax = y$ by a projection method means the following. Choose two sequences $(P_n)_{n=1}^\infty$ and $(R_n)_{n=1}^\infty$ of projections on X whose ranges (image spaces) $\operatorname{Im} P_n$ and $\operatorname{Im} R_n$ are finite-dimensional, $\dim \operatorname{Im} P_n = \dim \operatorname{Im} R_n = n$, say, and which converge strongly (i.e. pointwise) to the identity operator on X. Then replace the equation $Ax = y$ by the approximate equation

$$P_n A x^{(n)} = P_n y \quad (x^{(n)} \in \operatorname{Im} R_n). \tag{1}$$

One says that the projection method *converges* for the operator A and writes $A \in \Pi\{P_n, R_n\}$ in this case if there is an n_0 with the following property: the equations (1) have a unique solution $x^{(n)} \in \operatorname{Im} R_n$ for every $n \geq n_0$ and every $y \in X$, and $x^{(n)}$ converges in X to a solution x of the equation $Ax = y$.

For example, assume X is l^p, the Banach space of all sequences summable in the p th power, and A is given by an infinite matrix on l^p. If we denote by $P_n = R_n$ the projections on l^p given by

$$(x_1, x_2, x_3, \dots) \mapsto (x_1, x_2, \dots, x_n, 0, 0, \dots), \tag{2}$$

then the equation (1) is equivalent to the linear algebraic system whose matrix is the principal $n \times n$ section of A and whose right-hand side is constituted by the first n terms of the sequence y.

As another example, suppose $X = L^2(\Gamma)$ with some curve Γ in the plane and A is a linear and bounded integral operator on $L^2(\Gamma)$. Choose n "trial functions" $\varphi_1, \dots, \varphi_n$ and n "test functions" $\psi_1, \dots, \psi_n$ in $L^2(\Gamma)$, and denote by R_n and P_n the orthogonal projection of $L^2(\Gamma)$ onto the linear hull of $\varphi_1, \dots, \varphi_n$ and $\psi_1, \dots, \psi_n$, respectively. Then (1) requires finding a linear combination $x^{(n)} = x_1^{(n)}\varphi_1 + \dots + x_n^{(n)}\varphi_n$ of the trial functions such that the test equations $(Ax^{(n)}, \psi_j) = (y, \psi_j)$ are satisfied for $j = 1, \dots, n$. Projection methods of this type are often referred to as Galerkin methods.

Whether a given projection method converges for a given operator is in general a delicate problem. Standard texts on functional and numerical analysis usually cover self-adjoint operators and compact perturbations of the identity operator. However, operators of convolution type are in general neither self-adjoint nor of the form identity plus compact. This implies that approximation methods for convolution equations require new ideas and techniques and are therefore especially attractive. In the meanwhile the corresponding theory has grown up to an impressive edifice, a new significant top of which is the book by Hagen, Roch, and Silbermann.

1. The art of proving the convergence of approximation methods

For the sake of simplicity, let us consider projection methods with $P_n = R_n$. Denote by A_n the compression of A to $\operatorname{Im} P_n$, i.e. put $A_n = P_n A | \operatorname{Im} P_n$. The projection method thus gives us a sequence $(A_n)_{n=1}^{\infty}$ of approximating operators. This sequence is said to be *stable* if the operators A_n are invertible for all sufficiently large n, for $n \geq n_0$, say, and the norms of the inverses are uniformly bounded: $\sup_{n \geq n_0} \|A_n^{-1}\| < \infty$. In standard courses on numerical analysis we learn that $A \in \Pi\{P_n, P_n\}$ if and only if A is invertible and the sequence (A_n) is stable. Thus, everything is reduced to studying the stability of (A_n).

There are two cases in which the convergence of projection methods can be established almost straightforwardly. First, if X is a separable Hilbert space, A is a positive-definitive operator on X, and (P_n) is a sequence of orthogonal projections, then $(Ax, x) \geq \varepsilon \|x\|^2$, and consequently, $(P_n A P_n x, P_n x) = (A P_n x, P_n x) \geq \varepsilon \|P_n x\|^2$ for all $x \in X$. It follows that A_n is invertible for all n and that $\|A_n^{-1}\| \leq 1/\varepsilon$, which shows that $A \in \Pi\{P_n, P_n\}$. Secondly, in case $A = I + K$ with some compact operator K, the strong convergences $P_n \to I$ and $P_n^* \to I$ imply the uniform convergence $P_n K P_n \to K$. Hence, if $A = I + K$ is invertible, then so also is $A_n = I + P_n K P_n$ (on $\operatorname{Im} P_n$) for all sufficiently large n; and, moreover, since $\|(I + P_n K P_n)^{-1}\| \to \|(I + K)^{-1}\|$, it follows that the norms $\|A_n^{-1}\|$ are uniformly bounded. Consequently, $A \in \Pi\{P_n, P_n\}$ whenever A is invertible. By suitably modifying and combining the previous two arguments, one can prove the convergence of projection methods for a series of interesting operators.

But now suppose A is given on l^p $(1 < p < \infty)$ by an infinite Toeplitz matrix:

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3}$$

The function a defined on the complex unit circle $\mathbf{T}$ by $a(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ is referred to as the symbol of the operator induced by the matrix (3), and this operator is usually denoted by $T(a)$. Define P_n by (2) and abbreviate $P_n T(a) | \operatorname{Im} P_n$ to $T_n(a)$. The corresponding projection method is called the finite section method: it consists in replacing the infinite system with the matrix (3) by approximating systems whose matrices are

$$\begin{pmatrix} a_0 & \dots & a_{-(n-1)} \\ \dots & \dots & \dots \\ a_{n-1} & \dots & a_0 \end{pmatrix}.$$

Toeplitz operators are in general far away from being positive-definite or compactly perturbed identities. The study of the finite section method for Toeplitz operators was initiated in the early sixties by Baxter and Reich. Almost at the same time, Gohberg and Feldman embarked on this topic, and within only a few years they established a comprehensive theory for operators on l^p with continuous symbols and operators on l^2 with piecewise continuous symbols [2]. The basic idea of the Gohberg/Feldman approach is a clever combination and generalization of the arguments pointed out above: if $A = B + K + D$ where $B \in \Pi\{P_n, P_n\}$, K is compact, and D is of small norm, then $A \in \Pi\{P_n, P_n\}$ if only A is invertible. Suppose, for example, the symbol a of $T(a)$ is sufficiently smooth and $T(a)$ is invertible. Then a admits a so-called Wiener-Hopf factorization $a = a_- a_+$, which yields a representation

$T(a) = T(a_-)T(a_+)$. A simple trick shows that $B := T(a_+)T(a_-) \in \Pi\{P_n, P_n\}$. One can prove that $T(a) - B$ is compact. Consequently, $T(a) \in \Pi\{P_n, P_n\}$.

This “perturbation idea” was worked out and extended to other classes of convolution operators by several people in the seventies. An account of this development is given in Prössdorf and Silbermann’s book [4].

Nevertheless, some basic problems, mainly those pertaining to operators with discontinuous symbols or coefficients, remained open at that time. For instance, Verbitsky and Krupnik showed that if a is a piecewise continuous function with only a single jump, then the finite section method converges for $T(a)$ on l^p if and only if $T(a)$ itself and a certain associated (Toeplitz) operator $T(\tilde{a})$ are invertible. Thus, in more complicated situations the invertibility of $T(a)$ alone does not imply the convergence of the finite section method. However, all efforts to extend the Verbitsky/Krupnik result to piecewise continuous symbols with an arbitrary number of jumps failed. This problem was disposed of only after understanding how to apply Banach algebra techniques to projection methods.

2. What is spectral theory of approximation methods ?

Gelfand theory may be advantageously used to study invertibility in commutative Banach algebras. If a noncommutative Banach algebra has a nontrivial center, one may employ so-called local principles in order to reduce the question of whether an element is invertible to the same question for a family of (simpler) “local representatives”.

In the sixties, Simonenko realized that algebras of convolution operators have a nontrivial center modulo compact operators. Since an operator is Fredholm if and only if it is invertible modulo compact operators, one can therefore invoke local principles to establish Fredholm criteria for convolution operators.

In the early seventies, Kozak developed an analogous approach to projection methods and thus gave birth to what is nowadays called spectral theory of approximation methods. The basic idea of this theory is to construct a Banach algebra $\mathcal{B}$ with a nontrivial center such that the stability of the approximating sequence $(A_n)_{n=1}^{\infty}$ is equivalent to the invertibility of a certain element in $\mathcal{B}$. Suppose the operators A_n are $n \times n$ matrices, $A_n \in \mathcal{L}(\mathbf{C}^n)$. Put

$$\mathcal{B}^0 = \mathcal{L}(\mathbf{C}) \oplus \mathcal{L}(\mathbf{C}^2) \oplus \mathcal{L}(\mathbf{C}^3) \oplus \dots \tag{4}$$

and notice that

$$\mathcal{C} = \left\{ (C_n)_{n=1}^{\infty} \in \mathcal{B}^0 : \|C_n\| \to 0 \text{ as } n \to \infty \right\} \tag{5}$$

is an ideal of $\mathcal{B}^0$. A moment’s thought reveals that (A_n) is stable if and only if $(A_n)+\mathcal{C}$ is invertible in $\mathcal{B} := \mathcal{B}^0/\mathcal{C}$. On the basis of this simple construction and appropriate localization techniques, Kozak was able to prove remarkable convergence criteria for projection methods for multidimensional convolutions with continuous symbols.

Nevertheless, this approach failed for operators with discontinuous symbols, and so the development paused for almost a decade. The breakthrough came at the beginning of the eighties with Silbermann. He first considered the finite section method for Toeplitz operators. The product $T_n(a)T_n(b)$ of two finite Toeplitz matrices is in general not a Toeplitz matrix. Instead, one has the formula (which was first explicitly written down by Widom)

$$T_n(a)T_n(b) = T_n(ab) - P_n H(a)H(\tilde{b})P_n - Q_n H(\tilde{a})H(b)Q_n$$

where Q_n acts by the rule

$$Q_n : (x_1, x_2, x_3, \dots) \mapsto (x_n, x_{n-1}, \dots, x_1, 0, 0, \dots)$$

and $H(c), H(\tilde{c})$ stand for the Hankel matrices (operators)

$$H(c) = \begin{pmatrix} c_1 & c_2 & c_3 & \dots \\ c_2 & c_3 & \dots & \dots \\ c_3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \; H(\tilde{c}) = \begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & \dots \\ c_{-2} & c_{-3} & \dots & \dots \\ c_{-3} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Since Hankel operators with continuous symbols are compact, Silbermann replaced the ideal (5) by the set

$$\mathcal{J} = \Big\{(P_n K P_n + Q_n L Q_n + C_n)_{n=1}^\infty \in \mathcal{B}^0 : K \text{ and } L \text{ compact, } \|C_n\| \to 0\Big\}.$$

This is not an ideal in all of $\mathcal{B}^0$, but $\mathcal{B}^0$ contains a sufficiently large subalgebra $\mathcal{S}$ such that $\mathcal{J}$ is an ideal in $\mathcal{S}$. In particular, the sequences $(T_n(a))_{n=1}^\infty$ belong to $\mathcal{S}$. Finally, Silbermann associated two operators $W_1 := W_1\Big((A_n)_{n=1}^\infty\Big)$ and $W_2 := W_2\Big((A_n)_{n=1}^\infty\Big)$ with each sequence $(A_n) \in \mathcal{S}$ and proved the following theorem: if $(A_n) \in \mathcal{S}$, then (A_n) is stable if and only if the two operators W_1 and W_2 and the element $(A_n) + \mathcal{J} \in \mathcal{S}/\mathcal{J}$ are invertible. The point is that invertibility in $\mathcal{S}/\mathcal{J}$ can now again be studied with the help of local principles and that the structure of $\mathcal{S}/\mathcal{J}$ is much nicer than the structure of Kozak's algebra $\mathcal{B}^0/\mathcal{C}$. If $A_n = T_n(a)$ with some piecewise continuous function a, then W_1 is $T(a)$ itself, W_2 is the associated Toeplitz operator $T(\tilde{a})$, and $(A_n) + \mathcal{J}$ turns out to be automatically invertible if $T(a)$ is. Thus, Silbermann's approach implied that if a is an arbitrary piecewise continuous symbol, then $T(a) \in \Pi\{P_n, P_n\}$ if and only if both $T(a)$ and $T(\tilde{a})$ are invertible, and so solved a problem that had been open for many years at that time.

Spectral theory of approximation methods is the modification, extension, and generalization of the idea sketched in the preceding paragraph for the finite section method for Toeplitz operators to other approximation methods for other operators. Roughly speaking, one has to find the analogues of the ideal $\mathcal{J}$, of the algebra $\mathcal{S}$, and of the operators W_1, W_2 for the method under consideration.

The appearance of only two operators W_1, W_2 and of only one quotient algebra $\mathcal{S}/\mathcal{J}$ is a peculiarity of the finite section method for Toeplitz operators. In general, one is led to families $\{W_t\}_{t\in\Omega}$ of operators and $\{\mathcal{S}_\tau/\mathcal{J}_\tau\}_{\tau\in\Sigma}$ of algebras. Consequently, the results typically read as follows: the sequence (A_n) of approximating operators is stable if and only if certain operators W_t $(t \in \Omega)$ are invertible and certain additional conditions C_τ $(\tau \in \Sigma)$ are satisfied.

The books [1] and [5] summarize part of the development of spectral theory of projection methods up to the end of the eighties.

3. The operators considered in the book

Hagen, Roch, and Silbermann study operators belonging to the closed algebra of singular integral operators with piecewise continuous coefficients over arbitrary composed Lyapunov curves. It should be emphasized that the integration curves may have corners, self-intersections, or endpoints. The underlying spaces, in which also convergence of the approximate solutions is measured, are Lebesgue spaces with so-called power weights.

In this generality, even the Fredholm theory of the operators under consideration is highly nontrivial. This theory, which was worked out by Dynin, Gohberg, Krupnik, Duduchava, Plamenevski, Senitchkin, Costabel, Roch, Silbermann, and others, is presented in a very clear manner, with a lot of improvements and important supplements to known results. The exposition is based on Roch and Silbermann's earlier text [3]: after "localization" (a la Allan/Douglas) and "straightening", one arrives at a system of Mellin convolutions on the semi-axis.

Many other operators more or less directly related to singular integral operators are also studied. These include Toeplitz and Wiener-Hopf operators, Toeplitz plus Hankel operators, Fourier and Mellin convolutions, operators with Carleman shifts or complex conjugation.

4. Concrete approximation methods studied in the book

Dictated by the need of considering operators on fairly general curves, the authors focus attention on spline approximation methods. Thus, an approximate solution $x^{(n)}$ of $Ax = y$ is sought in a certain spline space. The notion of splines is used by the authors in a wide sense: splines are functions satisfying certain axioms, and specification of these axioms yields the classical smoothest piecewise polynomial splines, or piecewise polynomial splines with defect, or wavelets. A precise definition of splines would go beyond the scope of this review; for what follows, readers not familiar with splines may simply think of a spline as a piecewise constant function.

Once the form in which $x^{(n)}$ is sought has been specified, there are several possibilities of determining $x^{(n)}$ (e.g. of evaluating the n coefficients in the representation of $x^{(n)}$ as a linear combination of piecewise constant functions). The authors discuss the following methods.

Galerkin methods. Find $x^{(n)}$ so that $(Ax^{(n)}, \psi_j) = (y, \psi_j)$ for given test functions (splines) $\psi_1, \dots, \psi_n$.

Collocation methods. Determine $x^{(n)}$ by the requirement that $(Ax^{(n)})(\tau_j) = y(\tau_j)$ at given points $\tau_1, \dots, \tau_n$.

Qualocation methods. Compute the integral hidden in the scalar product $(.\,, .)$ of the Galerkin method by a quadrature formula $(.\,, .)_Q$, and hence, look for an $x^{(n)}$ such that $(Ax^{(n)}, \psi_j)_Q = (y, \psi_j)_Q$ for given test functions $\psi_1, \dots, \psi_n$.

Quadrature methods. Evaluate the singular (and, possibly, other) integrals contained in the operator A by appropriate quadrature formulas, and thus, replace the operator A by another ("discretized") operator A_D. Then apply a Galerkin or a collocation method to the equation $A_D x = y$.

Quadrocation methods. Do the same as in the case of a quadrature method, but solve the equation $A_D x = y$ by a qualocation method, i.e. determine $x^{(n)}$ by $(A_D x^{(n)}, \psi_j)_Q = (y, \psi_j)_Q$.

Notice that quadrocation methods as well as collocation quadrature methods are "fully discretized" methods.

For each of the methods listed, the authors establish convergence criteria of the kind mentioned at the end of Section 2 of this review.

5. The book itself

With this book, Hagen, Roch, and Silbermann finished a big stage in the development of spline approximation methods, ranging from Costabel and Stephan's pioneering papers in the middle of the eighties, through the important work by Prössdorf, Elschner, Rathsfeld, Schmidt, Chandler, Graham, Arnold, Wendland, Saranen, Sloan, and others, up to the recent investigations by Dahmen, Prössdorf, and Schneider devoted to wavelet approximations. Of course, the three authors themselves have been actively participating in this development, and many ideas and results of the book as well as the basic techniques employed are due to them. The theory they can present now is really round: on the basis of a unified approach they obtain results of a final nature.

Whether the book is easy to read depends on what the reader expects from the book. For novices who want to learn Banach algebra theory of approximation methods, the book is a useful guide on the long way they have to go. People who have been working in the field or in related areas, and thus have already gained some feeling for what is essential or not, will appreciate the book as an excellent source. However, those who have an equation and are consulting the book for advice on how to solve it approximately will run into trouble, because concrete recommendations are entirely missing, and both locating and decoding the relevant convergence result is sometimes no simple task.

I should also mention that only 6 of the 373 pages of the text are dedicated to numerical experiments. But this is okay, because the authors' aim is the rigorous foundation of several approximation methods. Numerically testing and comparing the different methods, developing fast algorithms, and implementing them on the computer is a great challenge for future work and the subject of another book.

Overall, as for the convergence and stability analysis of spline approximation methods for convolution and singular integral equations, I know of no even nearly comparable exposition of the field. The authors' approach is of fundamental importance; basic results of the book are new and of impressive depth. I believe the book will strongly influence further research into the topic and will be an indispensible source for a long time to come.

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